Math-1 (JAN, 2025)

Group-A

(1) (i) $\log_{\sqrt{3}} 27$

$= \log_{\sqrt{3}} 3^3$

$= 3 \log_{\sqrt{3}} 3$

$= 3 \log_{\sqrt{3}} \sqrt{3}^2$

$= 3 \times 2 \cdot \log_{\sqrt{3}} \sqrt{3}$

$= 3 \times 2$

$= 6$ Ans

(ii) Given,

$3x^2 - 5x + C = 0$

$\therefore \alpha + \beta = -\frac{b}{a} = \frac{5}{3}$

$\because \alpha = 1$

$\therefore \beta = \frac{5}{3} - \alpha = \frac{5}{3} - 1$

$= \frac{5-3}{3}$

$= 2/3$ Ans

(iii) 5th term in the expansion of $\left(1 - \frac{1}{x}\right)^{10}$

$\therefore t_5 = t_{4+1} = {}^{10}C_4 \cdot (1)^{10-4} \cdot \left(-\frac{1}{x}\right)^4$

$= {}^{10}C_4 \cdot 1^6 \cdot \frac{1}{x^4}$

$= 210 \times \frac{1}{x^4}$

$= \frac{210}{x^4}$ Ans

$\therefore {}^{10}C_4$

$= \frac{\lfloor 10}{\lfloor 4 \lfloor 10-4}$

$= \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot \cancel{\lfloor 6}}{4 \cdot 3 \cdot 2 \cdot 1 \cdot \cancel{\lfloor 6}}$

$= 210$

(iv) Here, $n = 10$ (even no.)

$\therefore$ middle term is $= \frac{10}{2} + 1 = 5 + 1 = 6$th term

$\therefore t_6 = t_{5+1} = {}^{10}C_5 \, (x)^{10-5} \cdot \left(\frac{1}{x}\right)^5$

$= 252 \times \cancel{x^5} \times \frac{1}{\cancel{x^5}}$

$= 252$ Ans

(v) i^6

$= i^2 \cdot i^2 \cdot i^2$

$= (\sqrt{-1})^2 \times (\sqrt{-1})^2 \times (\sqrt{-1})^2$

$= (-1) \times (-1) \times (-1)$

$= -1$ Ans

(vi) $i^6 + \omega^6$

$= i^2 \cdot i^2 \cdot i^2 + \omega^3 \cdot \omega^3$

$= (\sqrt{-1})^2 \times (\sqrt{-1})^2 \times (\sqrt{-1})^2 + (1)^3 \times (1)^3$

$= (-1) + 1$

$= \cancel{-1} + \cancel{1}$

$= 0$ Ans

(vii) $\dfrac{\cos 8^\circ + \sin 8^\circ}{\cos 8^\circ - \sin 8^\circ}$

$= \dfrac{1 + \dfrac{\sin 8^\circ}{\cos 8^\circ}}{1 - \dfrac{\sin 8^\circ}{\cos 8^\circ}}$

$= \dfrac{1 + \tan 8^\circ}{1 - \tan 8^\circ}$

$= \dfrac{\tan 45^\circ + \tan 8^\circ}{1 - \tan 45^\circ \tan 8^\circ}$

$= \tan(45^\circ + 8^\circ)$

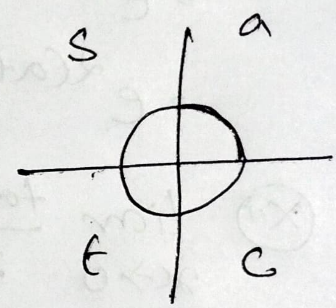
$= \tan 53^\circ$ Ans

(viii) $\cos 420^\circ$

$= \cos(5 \times 90^\circ - 30^\circ)$

$= \sin 30^\circ$

$= \frac{1}{2}$ Ans



(ix) $\because \vec{a} = 2i - 3j - 6k$

$\therefore |\vec{a}| = \sqrt{2^2 + 3^2 + 6^2}$

$= \sqrt{4 + 9 + 36}$

$= \sqrt{49}$

$= 7$ Ans

(x) $\vec{a} = 6i + 3j - 2k$ & $\vec{b} = 2i - 3j - 6k$

$\therefore \vec{a} \cdot \vec{b} = (6i + 3j - 2k) \cdot (2i - 3j - 6k)$

$= (6 \times 2) + [3 \times (-3)] + [(-2) \times (-6)]$

$= 12 - 9 + 12$

$= 24 - 9$

$= 15$ Ans

(xi) If $f(x) = e^{2x}$

$\therefore f(a) = e^{2a}$ & $f(b) = e^{2b}$

$\therefore f(a) \times f(b)$

$= e^{2a} \times e^{2b}$

$= e^{2a+2b}$ Ans

$= e^{2(a+b)}$ Ans

(xii) $\lim\limits_{x \to 0} \frac{\tan x}{x}$

$= \lim\limits_{x \to 0} \frac{\sin x}{x} \cdot \frac{1}{\lim\limits_{x \to 0} \cos x}$

$= 1 \cdot \frac{1}{1}$

$= 1$ Ans

(xiv) $y = \cancel{x}x^3 e^{2x}$

$\therefore \frac{dy}{dx} = \frac{d}{dx} x^3 e^{2x}$

$= x^3 \cdot \frac{d}{dx} e^{2x} + e^{2x} \cdot \frac{d}{dx} x^3$

$= x^3 \cdot e^{2x} \cdot 2 + e^{2x} \cdot 3x^2$

$= 2x^3 e^{2x} + 3x^2 e^{2x}$

$= 3x^2 e^{2x} + 2x^3 e^{2x}$ Ans

(xiii)

$\because y = a\cos x + b\sin x$

$\therefore \frac{dy}{dx} = a \cdot \frac{d}{dx} \cos x + b \cdot \frac{d}{dx} \sin x$

$= a \cdot (-\sin x) + b\cos x$

$= b\cos x - a\sin x$

$\therefore \frac{d^2y}{dx^2} = b \cdot \frac{d}{dx} \cos x - a \cdot \frac{d}{dx} \sin x$

$= b(-\sin x) - a\cos x$

$= -b\sin x - a\cos x$

$= -(b\sin x + a\cos x)$

$= -y$ Ans

(xv) $\because f(x) = 3x - x^2$

$\therefore f'(x) = 3 - 2x$

$\therefore f'(x) = 0$

$\therefore 3 - 2x = 0$

$\therefore -2x = -3$

$\therefore x = 3/2$

$\therefore f''(x) = -2 < 0$

$\therefore$ Max value

$\therefore f(x) = 3x - x^2$

$= 3 \times \frac{3}{2} - \left(\frac{3}{2}\right)^2$

$= \frac{9}{2} - \frac{9}{4}$

$= \frac{18-9}{4}$

$= \frac{9}{4}$ Ans

Group-B

(2) (a) Let,

$(r+1)^{th}$ term in the given expansion will contain $\frac{1}{x^2}$ or x^{-2}

$\therefore t_{r+1} = {}^{10}C_r \left(-\frac{1}{x}\right)^r$

$= (-1)^r x^{-r}\ {}^{10}C_r$

by condition,

$-r = -2 \therefore r = 2$

$\therefore$ co-efficient of $\frac{1}{x^2}$ in the expansion of $\left(1-\frac{1}{x}\right)^{10}$ is

$= (-1)^2 \cdot {}^{10}C_2 = \frac{\cancel{10}^{5} \cdot 9 \cdot \cancel{|8}}{\cancel{2} \cdot \cancel{|8}} = 45$ Ans

(b) Let, $x \neq 0$

$\therefore x + \frac{1}{x} = 2\cos\theta$

with $\theta = \pi/5$, then the 2 ~~solu~~ solution of x are $e^{i\theta}$ & $e^{-i\theta}$

$\therefore x^5 + \frac{1}{x^5} = e^{i5\theta} + e^{-i5\theta} = 2\cos 5\theta$

$\therefore \theta = \frac{\pi}{5}$, so $5\theta = \pi$

$\therefore x^5 + \frac{1}{x^5} = 2\cos 5\theta$

$= 2\cos\pi$

$= 2(-1)$

$= -2$ (proved)

③ ⓐ Given;

$\therefore a^x = bc$

$\therefore \log a^x = \log bc$

$\therefore x \log a = \log bc$

$\therefore x = \dfrac{\log bc}{\log a}$

$\therefore x = \log_a bc$

Similarly;

$y = \log_b ca$

&

$z = \log_c ab$

$\therefore$ L.H.S.

$= \dfrac{x}{x+1} + \dfrac{y}{y+1} + \dfrac{z}{z+1}$

$= \dfrac{\log_a bc}{\log_a bc + \log_a a} + \dfrac{\log_b ca}{\log_b ca + \log_b b} + \dfrac{\log_c ab}{\log_c ab + \log_c c}$

$= \dfrac{\log_a bc}{\log_a abc} + \dfrac{\log_b ca}{\log_b abc} + \dfrac{\log_c ab}{\log_c abc}$

$= \log_a bc \times \log_{abc} a + \log_b ca \times \log_{abc} b + \log_c ab \times \log_{abc} c$

$= \log_{abc} bc + \log_{abc} ca + \log_{abc} ab$

$= \log_{abc} a^2b^2c^2$

$= \log_{abc} (abc)^2$

$= 2\log_{abc} abc$

$= 2 \times 1 = 2$ (proved)

⑤ $\because 9x^2 - kx + 16 = 0$

Here; $a = 9$, $b = -k$, & $c = 16$

$\because$ roots are equal

$\therefore b^2 - 4ac = 0$

$\therefore (-k)^2 - 4 \times 9 \times 16 = 0$

$\therefore k^2 - 576 = 0$

$\therefore k^2 = 576$

$\therefore k = \sqrt{576}$

$\therefore k = \pm 24$ Ans

④ Given; $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$ & $\vec{b} = -2\hat{i} + 4\hat{j} - 2\hat{k}$

ⓐ $\vec{a} \cdot \vec{b}$

$= (\hat{i} + 2\hat{j} + 3\hat{k}) \cdot (-2\hat{i} + 4\hat{j} - 2\hat{k})$

$= (1)(-2) + (2)(4) + (3)(-2)$

$= -2 + 8 - 6$

$= \cancel{8} - \cancel{8}$

$= 0$

* After calculation of $\vec{a} \cdot \vec{b}$ we get $\vec{a} \cdot \vec{b} = 0$

So, these vectors are perpendicular to each other. Ans

ⓑ $\vec{a} \times \vec{b}$

$$= \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ -2 & 4 & -2 \end{vmatrix}$$

$= i\{2 \times (-2) - 3 \times 4\} - j\{(1) \times (-2) - 3 \times (-2)\} + k\{1 \times 4 - 2 \times (-2)\}$

$= \hat{i}(-4 - 12) - \hat{j}(-2 + 6) + k(4 + 4)$

$= -16\hat{i} - 4\hat{j} + 8k$ Ans

⑤ ⓐ Given, $a = 2i - j + 2k$

$b = -3i + j - 3k$

$\therefore a \times b = \begin{vmatrix} i & j & k \\ 2 & -1 & 2 \\ -3 & 1 & -3 \end{vmatrix}$

$= i(3-2) - j(-6+6) + k(2-3)$

$= i - k$

$\therefore |a \times b| = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\therefore$ Unit vector perpendicular

$= \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|} = \frac{i-k}{\sqrt{2}} = \frac{1}{\sqrt{2}}(i-k)$ Ans

ⓑ Given,

$F_1 = 14$, $F_2 = 7$, $F_3 = 21$

& $d_1 = 2i + 3j + 6k$

$d_2 = 6i + 3j - 2k$

$d_3 = -3i + 6j - 2k$

$\therefore$ For $F_1 \Rightarrow$

$|d_1| = \sqrt{4+9+36} = 7$

$\therefore \hat{d_1} = \frac{1}{7}(2i - 3j + 6k)$

$\therefore \vec{F_1} = k\hat{d_1}$

$\therefore \vec{F_1} = (4, -6, 12)$

Similarly

$F_2 = 6, 3, -2$

$F_3 = -9, 18, -6$

$\therefore$ Resultant force

$\therefore F_R = F_1 + F_2 + F_3$

$= (1, 15, 4)$ Ans

⑥ ⓐ.

L.H.S.

$= \sec\alpha + \tan\alpha$

$= \dfrac{1+\sin\alpha}{\cos\alpha}$

$= \dfrac{\sin^2\frac{\alpha}{2} + \cos^2\frac{\alpha}{2} + 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}}$

$= \dfrac{\left(\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}\right)^2}{\left(\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}\right)\left(\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right)}$

$= \dfrac{\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}}$

$= \dfrac{1+\tan\frac{\alpha}{2}}{1-\tan\frac{\alpha}{2}}$

$= \dfrac{\tan\frac{\pi}{4} + \tan\frac{\alpha}{2}}{1-\tan\frac{\pi}{4}\tan\frac{\alpha}{2}}$

$= \tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)$ (proved)

6 (b)

L.H.S.

$= \cos 20^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ$

$= \frac{1}{2} \cos 20^\circ \cos 40^\circ \cos 80^\circ$

$= \frac{1}{2 \cdot 2} (2\cos 20^\circ \cos 40^\circ) \cos 80^\circ$

$= \frac{1}{4} (\cos 60^\circ + \cos 20^\circ) \cos 80^\circ$

$= \frac{1}{4} \cdot \frac{1}{2} \cos 80^\circ + \frac{1}{4 \cdot 2} 2\cos 20^\circ \cos 80^\circ$

$= \frac{1}{8} \cos 80^\circ + \frac{1}{8} (\cos 100^\circ + \cos 60^\circ)$

$= \frac{1}{8} \cos 80^\circ + \frac{1}{8} \cos 100^\circ + \frac{1}{8} \cdot \frac{1}{2}$

$= \frac{1}{8} \cos 80^\circ + \frac{1}{8} \cos 100^\circ + \frac{1}{16}$

$= \frac{1}{8} \cos 80^\circ + \frac{1}{8} \cos (180^\circ - 80^\circ) + \frac{1}{16}$

$= \cancel{\frac{1}{8} \cos 80^\circ - \frac{1}{8} \cos 80^\circ} + \frac{1}{16}$

$= \frac{1}{16}$ = R.H.S. (proved)

7 (a)

$\lim_{x \to 0} \frac{e^{ax} - 1}{\sin bx}$

$= \lim_{x \to 0} \frac{a \cdot \frac{e^{ax} - 1}{ax}}{b \cdot \frac{\sin bx}{bx}}$

$= \frac{a}{b} \cdot \lim_{x \to 0} \frac{\frac{e^{ax} - 1}{ax}}{\frac{\sin bx}{bx}}$

$= \frac{a}{b} \frac{\lim_{x \to 0} \frac{e^{ax} - 1}{ax}}{\lim_{x \to 0} \frac{\sin bx}{bx}}$

$= \frac{a}{b} \times \frac{1}{1}$

$= \frac{a}{b}$ Ans

7 (b) Given;

$f(x) = \sqrt{x^2 - 7x + 10}$

$\therefore x^2 - 7x + 10 = (x-2)(x-5)$

For the square root

$(x-2)(x-5) \geq 0$

This happens when;

$x \leq 2$ or $x \geq 5$

So, Domain

$(-\infty, 2] \cup [5, \infty)$ Ans

8 (a) $\therefore x^y = e^{x-y}$

$\therefore \log x^y = \log e^{x-y}$

$\therefore y \log x = (x-y) \log e$

$\therefore y \log x = (x-y) \Rightarrow \boxed{y = \frac{x}{1+\log x}}$

$\therefore \frac{d}{dx}(y \log x) = \frac{d}{dx}(x-y)$

$\therefore y \cdot \frac{d}{dx} \log x + \log x \cdot \frac{d}{dx} \cdot y = \frac{d}{dx}(x) - \frac{dy}{dx}$

$\therefore y \times \frac{1}{x} + \log x \cdot \frac{dy}{dx} = 1 - \frac{dy}{dx}$

$\therefore \log x \cdot \frac{dy}{dx} + \frac{dy}{dx} = 1 - y/x$

$\therefore \frac{dy}{dx}(\log x + 1) = 1 - \frac{y}{x}$

$\therefore \frac{dy}{dx} = \frac{1 - \frac{y}{x}}{1 + \log x} = \frac{x-y}{x(1+\log x)} = \frac{x - \frac{x}{1+\log x}}{x(1+\log x)}$

$= \frac{\cancel{x} + x\log x - \cancel{x}}{x(1+\log x)^2}$

$= \frac{\cancel{x}\log x}{\cancel{x}(1+\log x)^2} = \frac{\log x}{(1+\log x)^2}$ Ans

8 (b) Given

$y = e^{a\sin^{-1}x}$

$\therefore \frac{dy}{dx} = e^{a\sin^{-1}x} \cdot \frac{a}{\sqrt{1-x^2}} \cdot 1$

$= \frac{ay}{\sqrt{1-x^2}}$

$\therefore y_1 = \frac{ay}{\sqrt{1-x^2}} \therefore y_1^2 = \frac{a^2y^2}{1-x^2}$

$\therefore y_1^2(1-x^2) = a^2y^2$

After differentiat both side;

$2y_1y_2(1-x^2) + y_1^2(-2x) = a^2 \cdot 2yy_1$

Now dividing both side by $2y_1$;

$(1-x^2)y_2 - xy_1 - a^2y = 0$ (proved)