# INVERSE TRIGONOMETRIC FUNCTION

## [FORMULAS]

① $\sin^{-1}x = \operatorname{cosec}^{-1}\frac{1}{x} = \cos^{-1}\sqrt{1-x^2} = \sec^{-1}\frac{1}{\sqrt{1-x^2}}$

$= \tan^{-1}\frac{x}{\sqrt{1-x^2}} = \cot^{-1}\frac{\sqrt{1-x^2}}{x}$

② $\cos^{-1}x = \sec^{-1}\frac{1}{x}$

③ $\tan^{-1}x = \cot^{-1}\frac{1}{x}$

④ $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$

⑤ $\operatorname{cosec}^{-1}x + \sec^{-1}x = \frac{\pi}{2}$

⑥ $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$

⑦ $\tan^{-1}x + \tan^{-1}y$

$= \tan^{-1}\frac{x+y}{1-xy}$

⑧ $\tan^{-1}x - \tan^{-1}y$

$= \tan^{-1}\frac{x-y}{1+xy}$

⑨ $\cot^{-1}x + \cot^{-1}y$

$= \cot^{-1}\frac{xy-1}{y+x}$

⑩ $\cot^{-1}x - \cot^{-1}y = \cot^{-1}\frac{xy+1}{y-x}$

⑪ $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1}\frac{x+y+z-xyz}{1-xy-yz-zx}$

⑫ $\sin^{-1}x + \sin^{-1}y = \sin^{-1}\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\}$

⑬ $\sin^{-1}x - \sin^{-1}y = \sin^{-1}\{x\sqrt{1-y^2} - y\sqrt{1-x^2}\}$

⑭ $\cos^{-1}x + \cos^{-1}y = \cos^{-1}\{xy - \sqrt{(1-x^2)(1-y^2)}\}$

⑮ $\cos^{-1}x - \cos^{-1}y = \cos^{-1}\{xy + \sqrt{(1-x^2)(1-y^2)}\}$

⑯ $2\sin^{-1}x = \sin^{-1}(2x\sqrt{1-x^2}) = \cos^{-1}(1-2x^2)$

(17) $2\cos^{-1}x = \cos^{-1}(2x^2-1)$

(18) $2\tan^{-1}x = \tan^{-1}\dfrac{2x}{1-x^2}$

$= \sin^{-1}\dfrac{2x}{1+x^2}$

$= \cos^{-1}\dfrac{1-x^2}{1+x^2}$

(19) $3\sin^{-1}x = \sin^{-1}(3x-4x^3)$

(20) $3\cos^{-1}x = \cos^{-1}(4x^3-3x)$

(21) $3\tan^{-1}x = \tan^{-1}\dfrac{3x-x^3}{1-3x^2}$

(22) $3\cot^{-1}x = \cot^{-1}\dfrac{x^3-3x}{3x^2-1}$

# INVERSE TRIGONOMETRIC FUNCTION

① Find the principle value of $\cot^{-1}(-1)$

⇒ Let's

$\cot^{-1}(-1) = \alpha$ where's $0 < \alpha < \pi$

↳ principle value

$\therefore \cot^{-1}(-1) = \alpha$

$\therefore \cot\alpha = -1 = \cot(\pi - \frac{\pi}{4}) = \cot\frac{3\pi}{4}$

$\therefore \alpha = \frac{3\pi}{4}$ Ans

② What is the general value of $\sin^{-1}(-\frac{1}{\sqrt{2}})$

⇒ Let's the general value is $\alpha$

where's $-\pi/2 \leq \alpha \leq \pi/2$

$\therefore \sin^{-1}(-\frac{1}{\sqrt{2}}) = \alpha$

$\therefore \sin\alpha = -\frac{1}{\sqrt{2}} = -\sin\frac{\pi}{4}$

$\therefore \alpha = -\frac{\pi}{4}$

$\therefore$ The general value is $= n\pi - (-1)^n \frac{\pi}{4}$ Ans

③ Evaluate $\sin[\cos^{-1}x + \operatorname{cosec}^{-1}\frac{1}{x}]$

⇒ $\sin(\cos^{-1}x + \operatorname{cosec}^{-1}\frac{1}{x})$

We know that $\sin^{-1}x = \operatorname{cosec}^{-1}\frac{1}{x}$

$\therefore \sin(\cos^{-1}x + \sin^{-1}x)$

$= \sin\pi/2$ $[\because \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}]$

$= \underline{1}$

④ $\sin\left[\tan^{-1}x + \tan^{-1}\frac{1}{x}\right]$

$= \sin\left[\tan^{-1}x + \cot^{-1}x\right]$ $\left[\because \tan^{-1}\frac{1}{x} = \cot^{-1}x\right]$

$= \sin\frac{\pi}{2}$ $\left[\because \tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}\right]$

$= 1$

⑤ $\sin\left\{\cos^{-1}\left(-\frac{1}{2}\right)\right\}$

$= \sin\left\{\cos^{-1}\left(\cos\frac{2\pi}{3}\right)\right\}$ $\left[\because -\frac{1}{2} = \frac{2\pi}{3}\right]$

$= \sin\frac{2\pi}{3}$ $\left[\because \cos^{-1}(\cos\theta) = \theta\right]$

$= \sin\left(\pi - \frac{\pi}{3}\right)$

$= \sin\pi/3$

$= \sin 60°$

$= \sqrt{3}/2$ Ans

⑥ $\sin\left(\cot^{-1}x\right)$

$\Rightarrow$ Let's $\cot^{-1}x = \theta$

$\therefore \cot\theta = x$

$\therefore \operatorname{cosec}\theta = \sqrt{1+x^2}$

$\therefore \sin\left(\cot^{-1}x\right)$

$= \sin\theta$

$= \frac{1}{\operatorname{cosec}\theta}$

$= \frac{1}{\sqrt{1+x^2}}$ Ans

⑦ Find the value of $\sec^2(\tan^{-1}2) + \csc^2(\cot^{-1}3)$

$\Rightarrow$ Let's $\tan^{-1}2 = \alpha$ & $\cot^{-1}3 = \beta$

$\therefore \tan\alpha = 2$  $\therefore \cot\beta = 3$

$\therefore \sec^2(\tan^{-1}2) + \csc^2(\cot^{-1}3)$

$= \sec^2\alpha + \csc^2\beta$

$= 1 + \tan^2\alpha + 1 + \cot^2\beta$

$= 1 + (2)^2 + 1 + (3)^2$

$= 1 + 4 + 1 + 9$

$= 15$ Ans

⑧ Find the principle value of $\sin^{-1}(\sin\frac{2\pi}{3})$

$\Rightarrow \sin^{-1}(\sin\frac{2\pi}{3})$

$= \sin^{-1}(\sin 120°)$

$= \sin^{-1}\{\sin(180° - 60°)\}$

$= \sin^{-1}\sin 60°$

$= 60°$

$= \pi/3$ Ans

⑨ If $\tan^{-1}(\frac{1}{2}\sec x) + \cot^{-1}(2\cos x) = \pi/3$ then $x = ?$

$\Rightarrow$ Let's $\tan^{-1}(\frac{1}{2}\sec x) = \theta$

$\therefore \frac{1}{2}\sec x = \tan\theta$

$\therefore \frac{1}{2\cos x} = \tan\theta$

$\therefore 2\cos x = \cot\theta$

Now,

$\therefore \theta + \cot^{-1}(\cot\theta) = \frac{\pi}{3}$

$\therefore \theta + \theta = \frac{\pi}{3}$

$\therefore 2\theta = \frac{\pi}{3}$

$\therefore \theta = \frac{\pi}{6}$ $\therefore 2\cos x = \cot\frac{\pi}{6}$

$\therefore 2\cos x = \sqrt{3}$

$\therefore \cos x = \frac{\sqrt{3}}{2}$

$\therefore \cos x = \cos\frac{\pi}{6}$ $\therefore x = \frac{\pi}{6}$ Ans

(10) prove that $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \frac{\pi}{4}$

=> L.H.S.

$= \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}$

$= \tan^{-1}\left(\frac{1/2 + 1/3}{1 - \frac{1}{2}\times\frac{1}{3}}\right)$ $\left[\because \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$

$= \tan^{-1}\left(\frac{5}{6}\times\frac{6}{5}\right)$

$= \tan^{-1}(1)$

$= \tan^{-1}\left(\tan\frac{\pi}{4}\right)$ $\left[\because 1 = \tan\frac{\pi}{4}\right]$

$= \pi/4$ = R.H.S. (proved)

(11) prove that $2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} = \frac{\pi}{4}$

=> L.H.S.

$= 2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7}$

$= \tan^{-1}\left(\frac{2\times 1/3}{1-(1/3)^2}\right) + \tan^{-1}\frac{1}{7}$

$= \tan^{-1}\frac{2}{3}\cdot\frac{9}{8} + \tan^{-1}\frac{1}{7}$

$= \tan^{-1}\frac{3}{4} + \tan^{-1}\frac{1}{7}$

$= \tan^{-1}\left(\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}\times\frac{1}{7}}\right)$

$= \tan^{-1}\left(\frac{25}{28}\times\frac{28}{25}\right)$

$= \tan^{-1}(1)$

$= \tan^{-1}\left(\tan\frac{\pi}{4}\right) = \frac{\pi}{4}$ = R.H.S. (proved)

(12) If $\cos^{-1}\left(\frac{x}{a}\right)+\cos^{-1}\left(\frac{y}{b}\right)=\theta$, then prove that

$$\frac{x^2}{a^2}-\frac{2xy}{ab}\cos\theta+\frac{y^2}{b^2}=\sin^2\theta$$

$\Rightarrow \because \cos^{-1}\left(\frac{x}{a}\right)+\cos^{-1}\left(\frac{y}{b}\right)=\theta$

or, $\cos^{-1}\left\{\frac{x}{a}\cdot\frac{y}{b}-\sqrt{\left(1-\frac{x^2}{a^2}\right)\left(1-\frac{y^2}{b^2}\right)}\right\}=\theta$

or, $\frac{xy}{ab}-\sqrt{\left(1-\frac{x^2}{a^2}\right)\left(1-\frac{y^2}{b^2}\right)}=\cos\theta$

or, $\frac{xy}{ab}-\cos\theta=\sqrt{\left(1-x^2/a^2\right)\left(1-y^2/b^2\right)}$

or, $\frac{x^2y^2}{a^2b^2}-\frac{2xy}{ab}+\cos^2\theta=\left(1-x^2/a^2\right)\left(1-y^2/b^2\right)$

or, $\cancel{\frac{x^2y^2}{a^2b^2}}-\frac{2xy}{ab}+\cos^2\theta=1-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\cancel{\frac{x^2y^2}{a^2b^2}}$

or, $\frac{x^2}{a^2}-\frac{2xy}{ab}+\frac{y^2}{b^2}=1-\cos^2\theta$

$\therefore \frac{x^2}{a^2}-\frac{2xy}{ab}+\frac{y^2}{b^2}=\sin^2\theta$ (proved)

(13) If $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=\pi$, then show that $x^2+y^2+z^2+2xyz=1$

$\Rightarrow \because \cos^{-1}x+\cos^{-1}y+\cos^{-1}z=\pi$

or, $\cos^{-1}x+\cos^{-1}y=\pi-\cos^{-1}z$

or, $\cos^{-1}\left\{xy-\sqrt{(1-x^2)(1-y^2)}\right\}=\pi-\theta$ [Here's $\cos^{-1}z=\theta$ $\therefore \cos\theta=z$]

or, $xy-\sqrt{(1-x^2)(1-y^2)}=\cos(\pi-\theta)$

or, $xy-\sqrt{(1-x^2)(1-y^2)}=-\cos\theta=-z$

or, $xy+z=\sqrt{(1-x^2)(1-y^2)}$

or, $(xy+z)^2 = (1-x^2)(1-y^2)$

or, $\cancel{x^2y^2}+z^2+2xyz = 1-x^2-y^2+\cancel{x^2y^2}$

or, $x^2+y^2+z^2+2xyz = 1$ (proved)

(14) If $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z = \frac{\pi}{2}$, then show that $xy+yz+zx=1$

$\Rightarrow \because \tan^{-1}x+\tan^{-1}y+\tan^{-1}z = \frac{\pi}{2}$

or, $\tan^{-1}x+\tan^{-1}y = \frac{\pi}{2}-\tan^{-1}z$

or, $\tan^{-1}x+\tan^{-1}y = \frac{\pi}{2}-\theta$ [Here, $\theta = \tan^{-1}z$
$\therefore \tan\theta = z$]

or, $\tan^{-1}\left(\frac{x+y}{1-xy}\right) = \frac{\pi}{2}-\theta$

or, $\frac{x+y}{1-xy} = \tan\left(\frac{\pi}{2}-\theta\right) = \cot\theta$

or, $\frac{x+y}{1-xy} = \frac{1}{z}$

or, $xz+yz = 1-xy$

or, $xz+yz+xy = 1$

$\therefore xy+yz+zx = 1$ (proved)

(15) $\tan^{-1}\frac{x}{y} - \tan^{-1}\frac{x-y}{x+y}$

$= \tan^{-1}\left[\dfrac{\frac{x}{y}-\frac{x-y}{x+y}}{1+\frac{x}{y}\cdot\frac{x-y}{x+y}}\right]$

$= \tan^{-1}\left[\frac{x^2+xy-xy+y^2}{xy+y^2+x^2-xy}\right] = \tan^{-1}\left(\frac{x^2+y^2}{x^2+y^2}\right)$

$= \tan^{-1}(1) = \frac{\pi}{4}$ Ans

(16) If two angles of a traingle are $\tan^{-1}2$ and $\tan^{-1}3$ then find the 3rd one.

$\Rightarrow$ Let's $A = \tan^{-1}2$ & $B = \tan^{-1}3$

$\therefore A + B = \tan^{-1}2 + \tan^{-1}3$

$= \tan^{-1}\left(\frac{2+3}{1-2\times3}\right)$

$= \tan^{-1}(-1)$

$= \tan^{-1}\left\{\tan\left(\pi - \frac{\pi}{4}\right)\right\}$

$= \tan^{-1}\left(\tan\frac{3\pi}{4}\right)$

$= \frac{3\pi}{4}$

Now, we know that sum of three angles of a traingle is $\pi$

$\therefore$ 3rd Angle is $= \pi - \frac{3\pi}{4} = \frac{\pi}{4}$ Ans

$\because A + B + C = \pi$

$\therefore A + B = \frac{3\pi}{4}$

$\therefore C = \pi - (A + B)$

(17) Show that $\sin\cos^{-1}\tan(\sec^{-1}x) = \sqrt{2-x^2}$

$\Rightarrow$ L.H.S.

$= \sin\cos^{-1}\tan(\sec^{-1}x)$

Let's $\sec^{-1}x = \theta$

$\therefore \sec\theta = x$

$\therefore \tan\theta = \sqrt{x^2-1}$

$= \sin\cos^{-1}\tan\theta$

$= \sin\cos^{-1}(\sqrt{x^2-1})$

$= \sin\alpha$ where's $\cos^{-1}\sqrt{x^2-1} = \alpha$

$\therefore \cos\alpha = \sqrt{x^2-1}$

$= \sqrt{2-x^2}$ Ans