INTEGRATION BY PARTIAL FRACTION

① $\int \frac{dx}{(x-2)(x-3)}$

$\Rightarrow$ Let;

$\therefore \frac{1}{(x-2)(x-3)} = \frac{A}{(x-2)} + \frac{B}{(x-3)}$ —— ①

$\therefore \frac{1}{(x-2)(x-3)} = \frac{A(x-3)+B(x-2)}{(x-2)(x-3)}$

$\therefore 1 = A(x-3)+B(x-2)$ —— ⑪

Now; put $x=3$ on eq^n ⑪, we get;

$\therefore 1 = A(\cancel{3}-\cancel{3}) + B(3-2)$

$\therefore 1 = A\times 0 + B\times 1$

$\boxed{\therefore B = 1}$

& put $x=2$ on eq^n ⑪, we get;

$\therefore 1 = A(2-3) + B(\cancel{2}-\cancel{2})$

$\therefore 1 = A(-1) + B\times 0$

$\therefore 1 = -A + 0$

$\boxed{\therefore A = -1}$

Now; value of A & B put on eq^n ①, we get;

$\therefore \frac{1}{(x-2)(x-3)} = \frac{-1}{(x-2)} + \frac{1}{(x-3)} = \frac{1}{(x-3)} - \frac{1}{(x-2)}$

$\therefore \int \frac{dx}{(x-2)(x-3)} = \int \frac{dx}{x-3} - \int \frac{dx}{x-2}$

$= \log|x-3| - \log|x-2| + c \quad \left[\because \int \frac{dx}{x-a} = \log|x-a|\right]$

$= \log\left|\frac{x-3}{x-2}\right| + c$ Ans $\left[\because \log m - \log n = \log\left(\frac{m}{n}\right)\right]$

(2) $\int \frac{dx}{x^2-4x-5}$

$\Rightarrow \because \frac{1}{x^2-4x-5} = \frac{1}{(x+1)(x-5)}$

Now;

Let; $\frac{1}{(x+1)(x-5)} = \frac{A}{x+1} + \frac{B}{x-5}$ —— (i)

$\therefore 1 = (x-5)A + (x+1)B$ —— (ii) [by simplifying eqn (i)]

And Now; put $x=5$ at eqn (ii), we get;

$\therefore 1 = (5-5)A + (5+1)B$

$\therefore 1 = 0 + 6B$

$\boxed{\therefore B = 1/6}$

& put $x=-1$ at eqn (ii), we get;

$\therefore 1 = (-1-5)A + (-1+1)B$

$\therefore 1 = (-6)A + 0$

$\boxed{\therefore A = -1/6}$

$\therefore$ Eqn (i), we get;

$\frac{1}{(x+1)(x-5)} = \frac{-1/6}{x+1} + \frac{1/6}{x-5} = \frac{1}{6(x-5)} - \frac{1}{6(x+1)}$

$\therefore \int \frac{dx}{x^2-4x-5}$

$= \int \frac{dx}{(x+1)(x-5)}$

$= \int \left[\frac{1}{(x+1)(x-5)}\right] dx$

$= \int \left[\frac{1}{6(x-5)} - \frac{1}{6(x+1)}\right] dx$

$= \int \frac{dx}{6(x-5)} - \int \frac{dx}{6(x+1)}$

$= \frac{1}{6}\int \frac{dx}{x-5} - \frac{1}{6}\int \frac{dx}{x+1}$

$= \frac{1}{6}\log|x-5| - \frac{1}{6}\log|x+1| + C \quad \left[\because \log|x+a| = \int \frac{dx}{x+a}\right]$

$= \frac{1}{6}\{\log|x-5| - \log|x+1|\} + C$

$= \frac{1}{6}\log\left|\frac{x-5}{x+1}\right| + C$ Ans

(3) $\int \frac{(2x+3)}{x^2-3x+2} dx$

$\therefore \int \frac{(2x+3)}{(x-2)(x-1)} dx$

$\therefore \frac{2x+3}{(x-2)(x-1)} = \frac{A}{(x-2)} + \frac{B}{(x-1)}$ —— (i)

Now; simplifying eqn (i) we get;

$\therefore 2x+3 = A(x-1) + B(x-2)$ —— (ii)

Now; put $x=1$, on eq^n (ii), we get;

$\therefore 2\times1+3 = A(1-1)+B(1-2)$

or, $5 = 0+(-1)B$

or, $5 = -B$

$\boxed{\therefore B=-5}$

& put $x=2$, on eq^n (ii), we get;

$\therefore 2\times2+3 = A(2-1)+B(2-2)$

$\therefore 7 = A(1)+0$

$\boxed{\therefore A=7}$

$\therefore$ eq^n (i) will become

$$\therefore \frac{2x+3}{x^2-3x+2} = \frac{7}{x-2} - \frac{5}{x-1}$$

$$\therefore \int \frac{2x+3}{x^2-3x+2}\,dx$$

$$= \int \left[\frac{7}{x-2} - \frac{5}{x-1}\right] dx$$

$$= \int \frac{7\,dx}{x-2} - \int \frac{5\,dx}{x-1}$$

$$= 7\int \frac{dx}{x-2} - 5\int \frac{dx}{x-1}$$

$$= 7\log|x-2| - 5\log|x-1| + C$$ Ans

(4) $\int \frac{\cos x\,dx}{\sin^2 x+4\sin x+3}$

Let; $\sin x = z$

$\therefore \cos x\,dx = dz$

$$\therefore \int \frac{dz}{z^2+4z+3}$$

$\therefore$ Now, Let,

$$\therefore \frac{1}{z^2+4z+3} = \frac{1}{(z+1)(z+3)} = \frac{A}{z+1} + \frac{B}{z+3} \quad \text{--- (i)}$$

$$\therefore 1 = A(z+3) + B(z+1) \quad \text{--- (ii)}$$

Now, put $z = -3$ on eqn (ii)

$$\therefore 1 = A(-3+3) + B(-3+1)$$

$$\therefore 1 = A \times 0 + B(-2)$$

$$\therefore 1 = -2B$$

$$\boxed{\therefore B = -1/2}$$

& put $z = -1$, on eqn (ii)

$$\therefore 1 = A(-1+3) + B(-1+1)$$

$$\therefore 1 = A(2) + 0$$

$$\boxed{\therefore A = 1/2}$$

$$\therefore \frac{1}{z^2+4z+3} = \frac{1/2}{z+1} + \frac{-1/2}{z+3} = \frac{1}{2(z+1)} - \frac{1}{2(z+3)}$$

$$\therefore \int \frac{dz}{z^2+4z+3} = \int \left[\frac{1}{2(z+1)} - \frac{1}{2(z+3)} \right] dz$$

$$= \int \frac{dz}{2(z+1)} - \int \frac{dz}{2(z+3)}$$

$$= \frac{1}{2} \log|z+1| - \frac{1}{2} \log|z+3| + C$$

$$= \frac{1}{2} \log\left|\frac{z+1}{z+3}\right| + C$$

$$= \frac{1}{2} \log\left|\frac{\sin x+1}{\sin x+3}\right| + C \quad \text{Ans}$$

⑤ $\int \frac{\cos x \, dx}{(\sin x - a)(\sin x - b)}$

Let; $\sin x = \theta$

$\therefore \cos x \, dx = d\theta$

$\Rightarrow \int \frac{d\theta}{(\theta - a)(\theta - b)}$

$\therefore \frac{1}{(\theta-a)(\theta-b)} = \frac{A}{(\theta-a)} + \frac{B}{(\theta-b)}$ — ⓘ

$\therefore 1 = A(\theta-b) + B(\theta-a)$ —— ⓘⓘ

Now; put $\theta = b$, on eqn ⓘⓘ

$\therefore 1 = A(b-b) + B(b-a)$

$\therefore 1 = B(b-a)$

$\therefore B = \frac{1}{b-a}$

& put $\theta = a$, on eqn ⓘⓘ

$\therefore 1 = A(a-b) + B(a-a)$

$\therefore 1 = A(a-b) + B \times 0$

$\therefore A = \frac{1}{a-b}$

$\therefore \frac{1}{(\theta-a)(\theta-b)} = \frac{1}{(a-b)(\theta-a)} + \frac{1}{(b-a)(\theta-b)}$

$= \frac{1}{(a-b)(\theta-a)} - \frac{1}{(a-b)(\theta-b)}$

$$\therefore \int \frac{d\theta}{(\theta-a)(\theta-b)}$$

$$= \int \left[\frac{1}{(a-b)(\theta-a)} - \frac{1}{(a-b)(\theta-b)} \right] d\theta$$

$$= \frac{1}{a-b} \int \frac{d\theta}{\theta-a} - \frac{1}{a-b} \int \frac{d\theta}{\theta-b}$$

$$= \frac{1}{a-b} \log|\theta-a| - \frac{1}{a-b} \log|\theta-b| + C$$

$$= \frac{1}{a-b} \log\left|\frac{\theta-a}{\theta-b}\right| + C$$

$$= \frac{1}{a-b} \log\left|\frac{\sin x - a}{\sin x - b}\right| + C$$ Ans

(6) $\int \frac{(x-1)\,dx}{(x+2)(x-3)}$

Let;

$$\frac{x-1}{(x+2)(x-3)} = \frac{A}{x+2} + \frac{B}{x-3} \quad \text{—— (i)}$$

$$\therefore x-1 = A(x-3) + B(x+2) \quad \text{—— (ii)}$$

Now; put $x=3$ on eq^n (ii), we get;

$\therefore 3-1 = A(3-3) + B(3+2)$

$\therefore 2 = 0 + B \times 5$

$\therefore 2 = 5B$

$\boxed{\therefore B = 2/5}$

$\therefore$ put $x=-2$ on eq^n (ii), we gets

$\therefore -2-1 = A(-2-3) + B(\cancel{-2+2})$

$\therefore -3 = A(-5) + B\times 0$

$\therefore -3 = -5A$

$\therefore 3 = 5A$

$\boxed{\therefore A = 3/5}$

$$\therefore \frac{(x-1)}{(x+2)(x-3)} = \frac{3/5}{x+2} + \frac{2/5}{x-3}$$

$$= \frac{3}{5(x+2)} + \frac{2}{5(x-3)}$$

$$\therefore \int \frac{(x-1)\,dx}{(x+2)(x-3)} = \int \frac{3dx}{5(x+2)} + \int \frac{2dx}{5(x-3)}$$

$$= \frac{3}{5}\int \frac{dx}{x+2} + \frac{2}{5}\int \frac{dx}{x-3}$$

$$= \frac{3}{5}\log|x+2| + \frac{2}{5}\log|x-3| + C \quad \text{Ans}$$

(7) $\int \frac{dx}{(x-1)^2(x-2)}$

Let; $\frac{1}{(x-1)^2(x-2)} = \frac{A}{x-2} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$ —— (i)

so, simplifying eq^n (i) we get;

$\therefore 1 = A(x-1)^2 + B(x-2)(x-1) + C(x-2)$ —— (ii)

Now; put $x=1$, on eq^n (ii)

$\therefore 1 = A(1-1)^2 + B(1-2)(1-1) + C(1-2)$

$\therefore 1 = A\times 0 + B\times 0 + C(-1)$

$\therefore 1 = -C$

$\boxed{\therefore C = -1}$

& put $x = 2$ on eqⁿ (ii), we get;

$\therefore 1 = A(2-1)^2 + B(\cancel{2-2})(2-1) + C(\cancel{2-2})$

$\therefore 1 = A(1)^2 + B\times0 + C\times0$

$\boxed{\therefore A = 1}$

Now; equating co-efficient of x^2 of eqⁿ (ii), we get;

$\therefore A + B = 0$

$\therefore B = -A = -1$

$\boxed{\therefore B = -1}$

$$\therefore \frac{1}{(x-1)^2(x-2)} = \frac{1}{x-2} - \frac{1}{x-1} - \frac{1}{(x-1)^2}$$

$$\therefore \int \frac{dx}{(x-1)^2(x-2)} = \int \frac{dx}{x-2} - \int \frac{dx}{x-1} - \int \frac{dx}{(x-1)^2}$$

$$= \log|x-2| - \log(x-1) - \left\{-\frac{1}{x-1}\right\} + C \quad \left[\because \int \frac{dx}{x^2} = -\frac{1}{x}\right]$$

$$= \log|x-2| - \log|x-1| + \frac{1}{x-1} + C$$

$$= \frac{1}{x-1} + \log\left|\frac{x-2}{x-1}\right| + C \text{ Ans}$$

(8) $\int \frac{(x+1)\,dx}{(x-1)(x+2)^2}$

Let; $\frac{x+1}{(x-1)(x+2)^2} = \frac{A}{(x-1)} + \frac{B}{(x+2)} + \frac{C}{(x+2)^2}$ — (i)

$$\therefore \frac{x+1}{(x-1)(x+2)^2} = \frac{A(x+2)^2 + B(x-1)(x+2) + C(x-1)}{(x-1)(x+2)^2}$$

$\therefore x+1 = A(x+2)^2 + B(x-1)(x+2) + C(x-1)$ — (ii)

Now; put $x = -2$ on eqⁿ (ii), we get;

$\therefore -2+1 = A(\cancel{-2+2})^2 + B(-2-1)(\cancel{-2+2}) + C(-2-1)$

$\therefore -1 = 0 + 0 + C(-3)$

$\therefore -1 = -3C$ $\boxed{\therefore C = 1/3}$

* put $x = 1$, on eqⁿ (ii)

$\therefore 1+1 = A(1+2)^2 + B(1-1)(1+2) + C(1-1)$

$\therefore 2 = A(3)^2 + 0 + 0$

$\therefore 2 = 9A$

$\boxed{\therefore A = 2/9}$

Now, equating co-efficient of x^2 of eqⁿ (ii)

$\therefore A + B = 0$

$\therefore B = -A = -2/9$

$\boxed{\therefore B = -2/9}$

$$\therefore \frac{(x+1)}{(x-1)(x+2)^2} = \frac{2/9}{x-1} - \frac{2/9}{x+2} + \frac{1/3}{(x+2)^2}$$

$$= \frac{2}{9(x-1)} - \frac{2}{9(x+2)} + \frac{1}{3(x+2)^2}$$

$$\therefore \int \frac{(x+1)\,dx}{(x-1)(x+2)^2} = \int \frac{2\,dx}{9(x-1)} - \int \frac{2\,dx}{9(x+2)} + \int \frac{dx}{3(x+2)^2}$$

$$= \frac{2}{9}\int \frac{dx}{x-1} - \frac{2}{9}\int \frac{dx}{x+2} + \frac{1}{3}\int \frac{dx}{(x+2)^2}$$

$$= \frac{2}{9}\log|x-1| - \frac{2}{9}\log|x+2| + \frac{1}{3}\left\{-\frac{1}{x+2}\right\} + c$$

$$= \frac{2}{9}\log|x-1| - \frac{2}{9}\log|x+2| - \frac{1}{3(x+2)} + c$$

Ans