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2ND ORDER

LINEAR ORDINARY DIFFERENTIAL EQUATION

MATHEMATICS - II

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2^{nd} order Linear ordinary Differential Equation with constant Co-efficient

formulas

If equation is $\Rightarrow \frac{d^2y}{dx^2} - 4\cdot\frac{dy}{dx} + 6y = 0$, then we can write is also-

- $(D^2y - 4Dy + 6y) = 0$

or, $(D^2 - 4D + 6)y = 0$

Because, $\boxed{\frac{d^2}{dx^2} = D^2}$ & $\boxed{\frac{d}{dx} = D}$

Finding A·E or Auxiliary Equation

If the equation is $(D^2 - 4D + 6)y = 0$

then just put $D = m$ then The A·E will become

$\boxed{\therefore m^2 - 4m + 6 = 0}$

The C.F. of the equation or required solution

C.F. = Complementary Function

$\Rightarrow$ First find roots (m_1 & m_2) from the A·E., then

(i) If m_1 & m_2 are ~~real~~ real & unequal

$\boxed{y = c_1 e^{m_1 x} + c_2 e^{m_2 x}}$

c_2 & c_a are the arbitrary constants & A & B as well.

(ii) If m_1 & m_2 are real & equal

$\boxed{y = (c_1 + c_2 x) e^{m_1 x}}$ or $\boxed{y = (c_1 + c_2 x) e^{m_2 x}}$

or,

$\boxed{y = (c_2 + c_1 x) e^{m_1 x}}$

(iii) If m_1 & m_2 are imaginary & $m_1 = \alpha + i\beta$ or $m_2 = \alpha - i\beta$

$\boxed{y = e^{\alpha x}(A\cos\beta x + B\sin\beta x)}$

Just put the values of m_1 & m_2 on the given equations as per the values of m_1 & m_2.

2ND ORDER LINEAR DIFFERENTIAL EQUATION
with Constant Co-efficient

① Find C.F. of $(D^2-2D+1)y = e^x$

$\Rightarrow$ Given; $(D^2-2D+1)y = e^x$

$\therefore$ A.E. $(D \Rightarrow m)$

$\therefore m^2-2m+1=0$

$\therefore (m-1)^2=0$

$\therefore (m-1)(m-1)=0$

$\therefore m_1=1$ & $m_2=1$ (So, roots are real & equal)

$\therefore$ The C.F. is

$\therefore y = (c_1+c_2x)e^{x\times 1}$ $[y = (c_1+c_2x)e^{m_1 x}]$

$\therefore y = (c_1+c_2x)e^x$ Ans

② Find C.F. of $\frac{d^2y}{dx^2} - 3\cdot\frac{dy}{dx} + 2y = e^x$

$\Rightarrow$ Given; $\frac{d^2y}{dx^2} - 3\cdot\frac{dy}{dx} + 2y = e^x$ or, $\underbrace{(D^2-3D+2)y}$ $= e^x$

Here; we put

$\frac{d^2y}{dx^2} = D^2y$ & $\frac{dy}{dx} = Dy$

or, $\frac{d^2}{dx^2} = D^2$ & $\frac{d}{dx} = D$

$\therefore$ A.E. is $(D \Rightarrow m)$

$\therefore m^2-3m+2=0$

or, $m^2-2m-m+2=0$

or, $m(m-2)-1(m-2)=0$

or, $(m-2)(m-1)=0$ (So, roots are real & unequal)

$\therefore m_1=2$ & $m_2=1$

$\therefore$ The C.F. is

$\therefore y = c_1e^{m_1x}+c_2e^{m_2x}$

$\therefore y = c_1e^{2x}+c_2e^x$ $[\because m_1=2$ & $m_2=1]$ Ans

③ Find C.F. of $\frac{d^2y}{dx^2} - 4y = 0$

$\Rightarrow$ Given, $\frac{d^2y}{dx^2} - 4y = 0$ or, $D^2y - 4y = 0$ or, $(D^2-4)y = 0$

$\therefore$ The A.E. ($D \Rightarrow m$)

$\therefore m^2 - 4 = 0$

or, $m^2 - 2^2 = 0$

or, $(m+2)(m-2) = 0$

$\therefore m_1 = -2$ & $m_2 = 2$ (So, roots are real & unequal)

$\therefore$ The C.F.

$\therefore y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$\therefore y = c_1 e^{-2x} + c_2 e^{2x}$ Ans

④ Find C.F. $\frac{d^2y}{dx^2} + 4 \cdot \frac{dy}{dx} + 4y = e^x$

$\Rightarrow$ Given, $\frac{d^2y}{dx^2} + 4 \cdot \frac{dy}{dx} + 4y = e^x$

or, $(D^2 + 4D + 4)y = e^x$ [By putting $\frac{d^2}{dx^2} = D^2$ & $\frac{d}{dx} = D$]

$\therefore$ The A.E. is

$\therefore m^2 + 4m + 4 = 0$

or, $(m)^2 + 2 \cdot m \cdot 2 + (2)^2 = 0$

or, $(m+2)^2 = 0$

or, $(m+2)(m+2) = 0$

$\therefore m_1 = -2$ & $m_2 = -2$ (real & equal)

$\therefore$ The C.F. is

$y = (c_1 + c_2 x) e^{m_1 x}$

$y = (c_1 + c_2 x) e^{-2x}$ Ans

⑤ Solve: $\frac{d^2y}{dx^2} - 6 \cdot \frac{dy}{dx} + 8y = 0$

$\Rightarrow$ Let's $\frac{d^2}{dx^2} = D^2$ & $\frac{d}{dx} = D$

$\therefore D^2y - 6Dy + 8y = 0$

$\therefore (D^2 - 6D + 8)y = 0$

Step 1

Now, the Auxiliary equation is (Let's put $D = m$)

$\therefore$ The A.E. is

$\therefore m^2 - 6m + 8 = 0$

or, $m^2 - 4m - 2m + 8 = 0$

or, $m(m-4) - 2(m-4) = 0$

or, $(m-4)(m-2) = 0$

Step 2

$\therefore m_1 = 4$ & $m_2 = 2$

Here's we can see roots are real & unequal

So, the required solution is

$\therefore y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

Step 3

Now's put the values of m_1 & m_2 on the above equation

$\therefore y = c_1 e^{4x} + c_2 e^{2x}$ Ans where's c_1 & c_2 are the arbitrary constants.

⑥ Solve: $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$

$\Rightarrow$ Given's $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$

$\therefore D^2y + Dy - 6y = 0$

$\therefore (D^2 + D - 6)y = 0$

$\therefore$ The Axiliary equation is

$\therefore m^2 + m - 6 = 0$

or, $m^2 + 3m - 2m - 6 = 0$

or, $m(m+3) - 2(m+3) = 0$

or, $(m+3)(m-2) = 0$

$\therefore m_1 = -3$ & $m_2 = 2$

$\therefore$ The roots are real & unequal

∴ The required solution is

$\therefore y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$\therefore y = c_1 e^{-3x} + c_2 e^{2x}$ Ans

where c_1 & c_2 are the arbitrary constants.

(7) Solve: $\frac{d^2y}{dx^2} - 9y = 0$

⇒ Given,

$\frac{d^2y}{dx^2} - 9y = 0$

or, $D^2y - 9y = 0$

$\therefore (D^2 - 9)y = 0$

∴ The Auxiliary equation is –

$\therefore m^2 - 9 = 0$

or, $(m)^2 - (3)^2 = 0$

or, $(m+3)(m-3) = 0$

$\therefore m_1 = -3$ & $m_2 = 3$

∴ The roots are real & unequal

∴ The required solution is –

$\therefore y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$\therefore y = c_1 e^{-3x} + c_2 e^{3x}$ Ans

(8) Solve: $\frac{d^2y}{dx^2} + 6 \cdot \frac{dy}{dx} + 9y = 0$

⇒ Given,

$\therefore \frac{d^2y}{dx^2} + 6 \cdot \frac{dy}{dx} + 9y = 0$

or, $D^2y + 6Dy + 9y = 0$

$\therefore (D^2 + 6D + 9)y = 0$

∴ The Auxiliary equation is –

$\therefore m^2 + 6m + 9 = 0$

or, $m^2+3m+3m+9=0$

or, $m(m+3)+3(m+3)=0$

$\therefore (m+3)(m+3)=0$

$\therefore m_1=-3$ & $m_2=-3$ [so, roots are real & equal]

$\therefore$ The required solution is –

$y=(C_1+C_2x)e^{m_1x}$

$y=(C_1+C_2x)e^{-3x}$ Ans [put $m_1=-3$]

⑨ Solve: $\frac{d^2y}{dx^2}-4\cdot\frac{dy}{dx}+4y=0$

$\Rightarrow$ Given; $\frac{d^2y}{dx^2}-4\cdot\frac{dy}{dx}+4y=0$

or, $D^2y-4Dy+4y=0$

$\therefore (D^2-4D+4)y=0$

$\therefore$ The Auxiliary equation is –

$\therefore m^2-4m+4=0$

or, $(m)^2-2\cdot m\cdot 2+(2)^2=0$

or, $(m-2)^2=0$

or, $(m-2)(m-2)=0$

$\therefore m_1=2$ & $m_2=2$

$\therefore$ roots are real & equal.

$\therefore$ The required solution is –

$\therefore y=(C_1+C_2x)e^{m_1x}$

$\therefore y=(C_1+C_2x)e^{2x}$ Ans

⑩ Solve: $4\cdot\frac{d^2y}{dx^2}+12\cdot\frac{dy}{dx}+5y=0$

$\Rightarrow$ Given; $4\cdot\frac{d^2y}{dx^2}+12\cdot\frac{dy}{dx}+5y=0$

or, $4D^2y+12Dy+5y=0$

$\therefore (4D^2+12D+5)y=0$

∴ The Auxiliary equation is —

∴ $4m^2+12m+9=0$

or, $(2m)^2+2\cdot 2m\cdot 3+(3)^2=0$

or, $(2m+3)^2=0$

or, $(2m+3)(2m+3)=0$

∴ $2m_1+3=0$ & $2m_2+3=0$

∴ $2m_1=-3$ ∴ $2m_2=-3$

∴ $m_1=-\frac{3}{2}$ ∴ $m_2=-\frac{3}{2}$

∴ The roots are real & ~~equal~~. equal.

∴ The required solution is —

∴ $y=(c_1+c_2x)e^{m_1x}$

∴ $y=(c_1+c_2x)e^{-\frac{3x}{2}}$ Ans

(11) solve: $\frac{d^3y}{dx^2}+9y=0$

⇒ Given, $\frac{d^2y}{dx^2}+9y=0$

or, $D^2y+9y=0$

∴ $(D^2+9)y=0$

∴ The Auxiliary equation is —

∴ $m^2+9=0$

or, $m^2=-9$

or, $m=\pm\sqrt{-9}=\pm\sqrt{-1}\cdot\sqrt{9}$

or, $m=\pm 3i$ [∵ $\sqrt{-1}=i$]

∴ $m_1=3i$ & $m_2=-3i$

so, the roots are imaginary.

Now, Let's compare m_1 with $\alpha+i\beta$ or m_2 with $\alpha-i\beta$

∴ $m_1=3i$

∴ $\alpha+i\beta=3i$

∴ $\alpha+i\beta=0+i\times 3$

∴ $\alpha=0$ & $\beta=3$

$\therefore$ The required solution is —

$\therefore y = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$

$\therefore y = e^{0\times x}(A\cos 3x + B\sin 3x)$

$\therefore y = A\cos 3x + B\sin 3x$ Ans where, A & B are the constants.

(12) solve: $(D^2+D+1)y=0$

$\Rightarrow$ Given, $(D^2+D+1)y=0$

$\therefore$ The Auxiliary equation is —

$\therefore m^2+m+1=0\ [a=1, b=1, c=1]$

or, $m = \dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

$= \dfrac{-1\pm\sqrt{1^2-4\cdot1\cdot1}}{2\cdot1}$

$= \dfrac{-1\pm\sqrt{1-4}}{2}$

$= \dfrac{-1\pm\sqrt{-3}}{2} = \dfrac{-1\pm\sqrt{-1}\cdot\sqrt{3}}{2}$

$= \dfrac{-1\pm i\sqrt{3}}{2}$

$\therefore m_1 = -\frac{1}{2}+\frac{\sqrt{3}}{2}i$ & $m_2 = -1-\frac{\sqrt{3}}{2}i$

$\therefore$ The roots are imaginary

Now, compare m_1 with $\alpha+i\beta$

$\therefore \alpha+i\beta = -\frac{1}{2}+\frac{\sqrt{3}}{2}i$

$\therefore \alpha = -\frac{1}{2}$ & $\beta = \frac{\sqrt{3}}{2}$

$\therefore$ The required solution is $y = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$

$= e^{-x/2}\left(A\cos\frac{\sqrt{3}x}{2} + B\sin\frac{\sqrt{3}x}{2}\right)$ Ans

(13) Solve: $y'' - 4y' + 8y = 0$

$\Rightarrow$ Given, $y'' - 4y' + 8y = 0$

or, $\frac{d^2y}{dx^2} - 4 \cdot \frac{dy}{dx} + 8y = 0$

or, $D^2y - 4Dy + 8y = 0$

$\therefore (D^2 - 4D + 8)y = 0$

$\therefore$ The Auxiliary equation is –

$\therefore m^2 - 4m + 8 = 0$

$\therefore m = \frac{4 \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot 8}}{2 \cdot 1}$

$= \frac{+4 \pm \sqrt{16 - 32}}{2}$

$= \frac{4 \pm \sqrt{-16}}{2}$

$= \frac{4 \pm \sqrt{-1}\sqrt{16}}{2}$

$= \frac{4 \pm 4i}{2}$

$\therefore m_1 = \frac{4}{2} + \frac{4}{2}i$ & $m_2 = 2 - 2i$

$= 2 + 2i$

Now, $\alpha + i\beta = 2 + 2i$

$\therefore \alpha = 2$ & $\beta = 2$

$\therefore$ The required solution is –

$\therefore y = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$

$\therefore y = e^{2x}(A\cos 2x + B\sin 2x)$ Ans