HOMOGENEOUS DIFFERENTIAL EQUATION

Steps to solve homogeneous diff. eqn —

step 1 : Let's $y = vx$

step 2 : Differential $y = vx$ w.r.t. x
then you will get $\frac{dy}{dx} = v + x \cdot \frac{dv}{dx}$

step 3 : Now's put $y = vx$ & $\frac{dy}{dx} = v + x \cdot \frac{dv}{dx}$ on the question or the given equation

step 4 : Try to separate $f(v)$ & $f(x)$

step 5 : After the separation, do integration to the both sides.

step 6 : Apply formulas & rules of integration

step 7 : After the formulazition again put $v = \frac{y}{x}$ & then you will get your answer.

HOMOGENEOUS DIFFERENTIAL EQUATION

① solve: $\frac{dy}{dx} = \frac{x-y}{x+y}$

Let; $y = vx$

Now differential both sides w.r.t. x, we get

$$\frac{dy}{dx} = v + x \cdot \frac{dv}{dx} \quad \left[\because \frac{d}{dx}(uv) = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}\right]$$

So, the given differential eqn reduces to —

$$\therefore \frac{dy}{dx} = \frac{x-y}{x+y}$$

or, $v + x \cdot \frac{dv}{dx} = \frac{x - vx}{x + vx} = \frac{1-v}{1+v}$

or, $x \cdot \frac{dv}{dx} = \frac{1-v}{1+v} - v$

or, $x \cdot \frac{dv}{dx} = \frac{1 - v - v - v^2}{1+v} = \frac{1-2v-v^2}{1+v}$

or, $\frac{1+v}{1-2v-v^2} dv = \frac{dx}{x}$

or, $\int \frac{1+v}{1-2v-v^2} dv = \int \frac{dx}{x}$

or, $-\frac{1}{2}\int \frac{d(1-2v-v^2)}{(1-2v-v^2)} = \int \frac{dx}{x}$

or, $-\frac{1}{2}\log(1-2v-v^2) = \log x + c$

$\therefore \log x\sqrt{1-2\left(\frac{y}{x}\right)-\left(\frac{y}{x}\right)^2} + c = 0 \quad \left[\because y = vx \quad \therefore v = \frac{y}{x}\right]$

$\therefore \log(x^2 - 2xy - y^2) + 2c = 0$ Ans

② Solve: $\frac{dy}{dx} = \frac{y(y+x)}{x(y-x)}$

$\Rightarrow$ Let; $y = vx$

$\therefore$ Differentiat both sides w.r.t. x, we get;

$\therefore \frac{dy}{dx} = v + x \cdot \frac{dv}{dx}$

$\therefore$ The above eqn becomes;

$\therefore v + x \cdot \frac{dv}{dx} = \frac{vx(vx+x)}{x(vx-x)}$ $\left[\because \frac{dy}{dx} = v + x \cdot \frac{dv}{dx} \text{ \& } y = vx\right]$

or, $v + x \cdot \frac{dv}{dx} = \frac{v\cancel{x^2}(v+1)}{\cancel{x^2}(v-1)}$

or, $v + x \cdot \frac{dv}{dx} = \frac{v^2+v}{v-1}$

or, $x \cdot \frac{dv}{dx} = \frac{v^2+v}{v-1} - v = \frac{\cancel{v^2}+v-\cancel{v^2}+v}{v-1} = \frac{2v}{v-1}$

or, $\frac{v-1}{2v} dv = \frac{dx}{x}$

or, $\left[\frac{v}{2v} - \frac{1}{2v}\right] dv = \frac{dx}{x}$

or, $\int \frac{1}{2} dv - \int \frac{1}{2v} dv = \int \frac{dx}{x}$

or, $\frac{1}{2} v - \frac{1}{2} \cdot \log v = \log x + c$

or, $\frac{1}{2} \cdot \frac{y}{x} - \frac{1}{2} \log\left(\frac{y}{x}\right) = \log x + c$ $\left[\because v = \frac{y}{x}\right]$

or, $\frac{y}{2x} - \log\sqrt{\frac{y}{x}} - \log x = c$

or, $\frac{y}{2x} - \log\sqrt{y} + \log\sqrt{x} - \log x = C$

or, $\frac{y}{2x} - \log\sqrt{y} + \log\left(\frac{\sqrt{x}}{x}\right) = C$

or, $\frac{y}{2x} - \log\sqrt{y} + \log\left(\frac{1}{\sqrt{x}}\right) = C$

or, $\frac{y}{2x} - \left[\log\sqrt{y} - \log\frac{1}{\sqrt{x}}\right] = C$

or, $\frac{y}{2x} - \log\frac{\sqrt{y}}{\frac{1}{\sqrt{x}}} = C$

or, $\frac{y}{2x} - \log\sqrt{xy} = C$

or, $\frac{y}{2x} = \log\sqrt{xy} + C$

$\therefore y = 2x(\log\sqrt{xy} + C)$ Ans

③ Solve: $\frac{dy}{dx} = \frac{x^2+y^2}{xy}$ or; $xy \cdot dy = (x^2+y^2)\,dx$

$\Rightarrow$ Let; $y = vx$

$\therefore$ Differentiat both sides w.r.t. x we get;

$\therefore \frac{dy}{dx} = v + x\cdot\frac{dv}{dx}$

So, the above eqn become;

$\therefore v + x\cdot\frac{dv}{dx} = \frac{x^2+v^2x^2}{x\times vx} = \frac{x^2+v^2x^2}{vx^2} = \frac{x^2(1+v^2)}{vx^2}$

or, $v + x\cdot\frac{dv}{dx} = \frac{1+v^2}{v}$

or, $x\cdot\frac{dv}{dx} = \frac{1+v^2}{v} - v = \frac{1+v^2-v^2}{v} = \frac{1}{v}$

or, $\int v\cdot dv = \int\frac{dx}{x}$

or, $\frac{1}{2}v^2 = \log|x| + C$

or, $\frac{1}{2}\left(\frac{y}{x}\right)^2 = \log x + c$

$\therefore \frac{y^2}{2x^2} = \log x + c$ Ans

④ solve: $(x^2+y^2)dx - 2xy\,dy = 0$

$\Rightarrow \because (x^2+y^2)dx - 2xy\,dy = 0$

$\therefore 2xy\,dy = (x^2+y^2)dx$

$\therefore \frac{dy}{dx} = \frac{x^2+y^2}{2xy}$

Let; $y = vx$

Now; differentiat both sides w.r.t. x we get;

$\therefore \cancel{\frac{dy}{dx} = x + x\frac{dy}{dx}}$

$\therefore \frac{dy}{dx} = v + x\cdot\frac{dv}{dx}$

Now; put on the above eqn —

$\therefore v + x\cdot\frac{dv}{dx} = \frac{x^2+v^2x^2}{2\times x\times vx} = \frac{x^2(1+v^2)}{2x^2v} = \frac{1+v^2}{2v}$

or, $x\cdot\frac{dv}{dx} = \frac{1+v^2}{2v} - v = \frac{1+v^2-2v^2}{2v} = \frac{1-v^2}{2v}$

or, $\int \frac{2v}{1-v^2}\,dv = \int \frac{dx}{x}$

Now; Let;

$1-v^2 = \theta$

$\therefore -2v\,dv = d\theta$

$\therefore 2v\,dv = -d\theta$

or, $\int -\frac{d\theta}{\theta} = \int \frac{dx}{x}$

or, $-\int \frac{d\theta}{\theta} = \int \frac{dx}{x}$

or, $-\log|\theta| = \log|x| + \log c$

or, $-\log|1-v^2| = \log|x| + \log|c|$

or, $\log\left|\frac{1}{1-v^2}\right| = \log|x \cdot c|$

or, $\frac{1}{1-v^2} = xc$

or, $\frac{1}{1-\frac{y^2}{x^2}} = xc$

or, $\frac{x^2}{x^2-y^2} = xc$

$\therefore xc(x^2-y^2) = x^2$ Ans

⑤ Solve: $(x^3+y^3)dx - xy^2dy = 0$

$\Rightarrow \because (x^3+y^3)dx = xy^2dy$

$\Rightarrow xy^2dy = (x^3+y^3)dx$

$\therefore \frac{dy}{dx} = \frac{x^3+y^3}{xy^2}$

Let; $y = vx$

$\therefore \frac{dy}{dx} = v + x\cdot\frac{dv}{dx}$

Now; put on the above eq^n —

$\therefore v + x\cdot\frac{dv}{dx} = \frac{x^3+v^3x^3}{x\cdot v^2x^2} = \frac{x^3(1+v^3)}{x^3v^2} = \frac{1+v^3}{v^2}$

or, $x\cdot\frac{dv}{dx} = \frac{1+v^3}{v^2} - v = \frac{1+v^3-v^3}{v^2} = \frac{1}{v^2}$

or, $\int v^2dv = \int\frac{dx}{x}$

or, $\frac{1}{3}v^3 = \log x + c$

or, $\frac{1}{3}\cdot\left(\frac{y}{x}\right)^3 = \log x + c$

$\therefore \frac{y^3}{3x^3} = \log x + c$ Ans

⑥ Solve: $\frac{dy}{dx} = \frac{y}{x}\left(\log\frac{y}{x} + 1\right)$

$\Rightarrow$ Let; $y = vx$

$\therefore \frac{dy}{dx} = v + x\cdot\frac{dv}{dx}$

Now; put on the above eqn —

$\therefore \frac{dy}{dx} = \frac{y}{x}\left(\log\frac{y}{x} + 1\right)$

or, $v + x\cdot\frac{dv}{dx} = \frac{vx}{x}\left(\log\frac{vx}{x} + 1\right)$

or, $\cancel{v} + x\cdot\frac{dv}{dx} = v(\log v + 1) = v\log v + \cancel{v}$

or, $x\cdot\frac{dv}{dx} = v\log v$

or, $\int\frac{dv}{v\log v} = \int\frac{dx}{x}$

Let; $\log v = \theta$

$\therefore \frac{1}{v}dv = d\theta$

or, $\int\frac{d\theta}{\theta} = \int\frac{dx}{x}$

or, $\log\theta = \log x + \log c$

or, $\log(\log v) = \log x + \log c$

or, $\log\left(\log\frac{y}{x}\right) = \log(xc)$

$\therefore \log\frac{y}{x} = xc$ Ans

⑦ Solve: $\frac{dy}{dx} = \frac{y(2y-x)}{x(2y+x)}$ when; $x=1$ & $y=1$

$\Rightarrow$ Let; $y = vx$

$\therefore \frac{dy}{dx} = v + x \cdot \frac{dv}{dx}$

Now; put on the above eqn —

$\therefore v + x \cdot \frac{dv}{dx} = \frac{vx(2 \cdot vx - x)}{x(2 \cdot vx + x)} = \frac{vx^2(2v-1)}{x^2(2v+1)}$

or, $v + x \cdot \frac{dv}{dx} = \frac{v(2v-1)}{(2v+1)}$

or, $x \cdot \frac{dv}{dx} = \frac{2v^2 - v}{2v+1} - v = \frac{\cancel{2v^2} - v - \cancel{2v^2} - v}{2v+1} = \frac{-2v}{2v+1}$

or, $\frac{2v+1}{2v} dv = -\frac{dx}{x}$

or, $dv + \frac{1}{2v} dv = -\frac{dx}{x}$

or, $\int dv + \int \frac{1}{2v} dv = -\int \frac{dx}{x}$

or, $v + \frac{1}{2} \log v = -\log x + c$

$\therefore \frac{y}{x} + \frac{1}{2} \log \frac{y}{x} = -\log x + c$

$\therefore \frac{y}{x} + \frac{1}{2} \log \frac{y}{x} + \log x = c$

put $x = 1$ & $y = 1$, we get;

$c = \frac{1}{1} + \frac{1}{2} \log \frac{1}{1} + \log 1 = 1 + 0 + 0 = 1$

$\therefore$ The soln is $\Rightarrow$

$\therefore \frac{y}{x} + \frac{1}{2} \log \frac{y}{x} + \log x = 1$ Ans

⑧ Solve: $\frac{dy}{dx} = \frac{y}{x} + \tan\left(\frac{y}{x}\right)$ when $y = \frac{\pi}{2}$ & $x = 1$

$\Rightarrow$ Let; $y = vx$

$\therefore \frac{dy}{dx} = v + x \cdot \frac{dv}{dx}$

Now; put on the above eqn —

$\therefore v + x \cdot \frac{dv}{dx} = \frac{vx}{x} + \tan\left(\frac{vx}{x}\right)$

or, $\cancel{v} + x \cdot \frac{dv}{dx} = \cancel{v} + \tan v$

or, $x \cdot \frac{dv}{dx} = \tan v$

or, $\frac{dv}{\tan v} = \frac{dx}{x}$

or, $\int \cot v \, dv = \int \frac{dx}{x}$

or, $\log \sin v = \log x + \log c$

or, $\sin v = xc$

or, $\sin\left(\frac{y}{x}\right) = x \cdot c$

Now; $\because y = \frac{\pi}{2}, x = 1$

$\therefore \sin\left(\frac{\pi/2}{1}\right) = 1 \cdot c$

$\therefore c = \sin\frac{\pi}{2} = 1$

$\therefore$ The soln is $\Rightarrow$

$\therefore \sin\frac{y}{x} = x \cdot 1$

$\therefore \sin\frac{y}{x} = x$ Ans