2^{nd} order Linear, Ordinary Differential Equations with constant co-efficients & method of solving Particular Integral (P.I.)

Some usefull expansion

① $(1-D)^{-1} = 1+D+D^2+D^3+\cdots\alpha$

② $(1+D)^{-1} = 1-D+D^2-D^3+\cdots\alpha$

Finding P.I. [When X is of the form x^m]

$$\therefore P.I. = \frac{x^m}{f(D)}$$

We can express $f(D)$ as $(1 \pm g(D))$.

So, $\frac{1}{f(D)}$ will be $[1 \pm g(D)]^{-1}$

Numericals;

① Find P.I. (Particular Integral) of the given expressions:

ⓐ $\frac{d^2y}{dx^2} - 2\cdot\frac{dy}{dx} + y = x$

$\therefore D^2y - 2Dy + y = x$ $\left[\because \frac{d^2}{dx^2} = D^2 \text{ \& } \frac{d}{dx} = D\right]$

$\therefore (D^2 - 2D + 1)y = x$

$$\therefore \text{P.I. for } x = \frac{x}{(D^2-2D+1)}$$

$= (D^2-2D+1)^{-1}x$

$= \{1+(D^2-2D)\}^{-1}x$

$= (1-(D^2-2D)+\cdots)x$ $[\because (1+D)^{-1} = 1-D+D^2\cdots]$

$= (1+2D\cdots)x$

$= x+2Dx = x+2\cdot\frac{d}{dx}(x) = x+2$ Ans

(6) $(D^2+1)y = x^3$

$\therefore$ The P.I. of x^3

$$= \frac{x^3}{(D^2+1)}$$

$$= (1+D^2)^{-1}x^3$$

$$= (1-D^2+D^4\dots\dots)x^3$$

$$= x^3 - D^2x^3$$

$$= x^3 - \frac{d^2}{dx^2}x^3$$

$$= x^3 - \frac{d}{dx}\left(\frac{d}{dx}x^3\right)$$

$$= x^3 - \frac{d}{dx}3x^2$$

$$= x^3 - 3\cdot 2\cdot x$$

$$= x^3 - 6x$$

(2) Solve: $y'' + 2y' + y = x$

$\Rightarrow$ Given; $y'' + 2y' + y = x$

$$\therefore \frac{d^2y}{dx^2} + 2\cdot\frac{dy}{dx} + y = x$$

$$\therefore D^2y + 2Dy + y = x$$

$$\therefore (D^2+2D+1)y = x$$

$\therefore$ The Auxilliary equation will be $(D=m)$

$$\therefore m^2+2m+1=0$$

$$\therefore (m+1)^2=0$$

$$\therefore (m+1)(m+1)=0 \quad (\because \text{roots are real \& equal})$$

$$\therefore m_1=-1, \ \& \ m_2=-1$$

$\therefore$ The C.F. is $= (A+xB)e^{-x}$

$\therefore$ The P.I. of x

$$= \frac{x}{(D^2+2D+1)}$$

$$= (1+D^2+2D)^{-1}x$$

$$= \{1+(D^2+2D)\}^{-1}x$$

$$= (1-2D\cdots)x$$

$$= x-2Dx$$

$$= x-2$$

$\therefore$ General solution is

$$y = C.F. + P.I.$$

$$y = (A+Bx)e^{-x} + x - 2 \quad \text{Ans}$$

③ Solve: $\frac{d^2y}{dx^2} + 4y = x^2$

$\Rightarrow$ Given: $\frac{d^2y}{dx^2} + 4y = x^2$

$$\therefore D^2y + 4y = x^2$$

$$\therefore (D^2+4)y = x^2$$

$\therefore$ The A.E. is $\Rightarrow m^2+4=0$

$$\therefore m^2 = -4$$

$$\therefore m = \pm\sqrt{-4}$$

$$\therefore m_1 = +\sqrt{-4} \;\&\; m_2 = -\sqrt{-4}$$

$$= +\sqrt{-1}.\sqrt{4}$$

$= +2i$ (roots are imaginary)

$= 0+2i$ $(A+iB)$

So, here: $A=0$ & $B=2$

$\therefore$ The C.F. is $y = e^{0.x}(A\cos 2x + B\sin 2x)$

$$= A\cos 2x + B\sin 2x$$

$\therefore$ The P.I. of x^2

$$= \frac{x^2}{(D^2+4)}$$

$$= \frac{x^2}{4\left(1+\frac{D^2}{4}\right)}$$

$$= \frac{1}{4}\left(1+\frac{D^2}{4}\right)^{-1} x^2$$

$$= \frac{1}{4}\left(1-\frac{D^2}{4}\cdots\right) x^2$$

$$= \frac{1}{4}\left(x^2-\frac{1}{4}D^2x^2\right)$$

$\therefore D^2x^2$

$= \frac{d}{dx}\left(\frac{d}{dx}x^2\right)$

$= \frac{d}{dx}2x$

$= 2$

$$= \frac{1}{4}\left(x^2-\frac{1}{4}\cdot 2\right)$$

$$= \frac{1}{4}\left(x^2-\frac{1}{2}\right)$$

$\therefore$ The general solution is

$\therefore y = C.F. + P.I.$

$\therefore y = (A\cos 2x + B\sin 2x) + \frac{1}{4}\left(x^2-\frac{1}{2}\right)$ Ans

(4) solve: $\frac{d^2y}{dx^2} + y = x^3$

$\Rightarrow$ Given: $\frac{d^2y}{dx^2} + y = x^3$

$\therefore (D^2+1)y = x^3$

$\therefore$ The A.E. is $\Rightarrow m^2+1=0$

$\therefore m^2 = -1$

$\therefore m = \pm\sqrt{-1}$

$\therefore m_1 = +\sqrt{-1}$ & $m_2 = -\sqrt{-1}$

$\therefore m_1 = +i$ (roots are imaginary)

$\therefore m_1 = 0 + 1\times i$ $(A+iB)$

$\therefore A = 0$ & $B = 1$

$\therefore$ The C.F. is

$$y = e^{0.x}(A\cos x + B\sin x)$$

$$y = A\cos x + B\sin x$$

$\therefore$ The P.I. of x^3

$$= \frac{x^3}{D^2+1}$$

$$= (1+D^2)^{-1}x^3$$

$$= (1-D^2+D^4 \ldots)x^3$$

$$= (1-D^2)x^3$$

$$= x^3 - D^2x^3$$

$$= x^3 - 6x$$

$\therefore D^2x^3$

$$= \frac{d}{dx}\left(\frac{d}{dx}x^3\right)$$

$$= \frac{d}{dx}(3x^2)$$

$$= 3.\frac{d}{dx}x^2$$

$$= 3.2x = 6x$$

$\therefore$ The general solution is

$$y = C.F. + P.I.$$

$$= A\cos x + B\sin x + x^3 - 6x \text{ Ans}$$

⑤ solve: $(D^2-3D+2)y = 3x$

$\Rightarrow$ Given: $(D^2-3D+2)y = 3x$

$\therefore$ The A.E. is

$\therefore m^2 - 3m + 2 = 0$

or, $m^2 - 2m - m + 2 = 0$

or, $m(m-2) - 1(m-2) = 0$

$\therefore (m-2)(m-1) = 0$

$\therefore m_1 = 2$ & $m_2 = 1$ (roots are real but unequal)

$\therefore$ The C.F. is $y = C_1e^{2x} + C_2e^{x}$

$\therefore$ The P.I.

$$= \frac{3x}{D^2 - 3D + 2}$$

$$= 3 \cdot \frac{1}{2\left(1 + \frac{D^2 - 3D}{2}\right)} x$$

$$= \frac{3}{2}\left(1 + \frac{D^2 - 3D}{2}\right)^{-1} x$$

$$= \frac{3}{2}\left(1 - \frac{D^2 - 3D}{2} + \frac{(D^2 - 3D)^2}{4} \cdots\right) x$$

$$= \frac{3}{2}\left(1 + \frac{3D}{2} - \frac{D^2}{2} \cdots\right) x$$

$$= \frac{3}{2}\left(x + \frac{3}{2} Dx\right)$$

$$= \frac{3}{2}\left(x + \frac{3}{2}\right)$$

$\therefore$ The general solution is

$\therefore y = C.F. + P.I.$

$\therefore y = c_1 e^{2x} + c_2 e^{x} + \frac{3}{2}\left(x + \frac{3}{2}\right)$ Ans