2^{nd} Order Linear, Ordinary Differential Equations with constant co-efficient and method of solving Particular Integral (PART-2)

When X is e^{ax}, where a is any constant than:

$$P.I. = \frac{X}{f(D)} = \frac{e^{ax}}{f(a)}$$

Provided $f(a) \neq 0$ i.e. we have to put $D = a$ in $f(d)$ and P.I. will be calculated accordingly.

Numericals:

① Find P.I. of

ⓐ $\frac{d^2y}{dx^2} - 5 \cdot \frac{dy}{dx} + 6y = e^{4x}$

$\therefore (D^2 - 5D + 6)y = e^{4x}$ $\left[\because \frac{d^2}{dx^2} = D^2 \ \& \ \frac{d}{dx} = D\right]$

$\therefore (D-3)(D-2)y = e^{4x}$

$$\therefore P.I. = \frac{e^{4x}}{(D-3)(D-2)}$$

$$= \frac{1}{D^2 - 5D + 6} e^{4x}$$

put $D = 4$, we get

$= 4^2 - 5 \times 4 + 6 = 2$

$$\therefore P.I. = \frac{e^{4x}}{2} \text{ Ans}$$

ⓑ $\frac{d^2y}{dx^2} + y = e^{2x}$

$\therefore (D^2 + 1)y = e^{2x}$

$\therefore P.I. = \dfrac{e^{2x}}{D^2+1}$

$= \dfrac{1}{D^2+1} e^{2x}$

$= \dfrac{1}{2^2+1^2} e^{2x}$ [Put $D=2$]

$= \dfrac{e^{2x}}{5}$ Ans

(c) $(D^2+5D+6)y = e^{-x}$

$\therefore P.I. = \dfrac{e^{-x}}{D^2+5D+6}$

$= \dfrac{1}{(-1)^2+5\times(-1)+6} e^{-x}$ [Put $D=-1$]

$= \dfrac{e^{-x}}{2}$ Ans

(d) $\dfrac{d^2y}{dx^2} - 4\cdot\dfrac{dy}{dx} + 3y = e^{2x}$

$\therefore (D^2-4D+3)y = e^{2x}$

$\therefore P.I. = \dfrac{e^{2x}}{D^2-4D+3}$

$= \dfrac{e^{2x}}{2^2-4\times2+3}$

$= \dfrac{e^{2x}}{-1}$

$= -e^{2x}$ Ans

② Solve: $(D^2+3D+2)y = e^{2x}$

$\Rightarrow$ Given;

$(D^2+3D+2)y = e^{2x}$

$\therefore$ The auxiliary equation is $(D=m)$

$\therefore m^2+3m+2=0$

or, $m^2+2m+m+2=0$

or, $m(m+2)+1(m+2)=0$

$\therefore (m+2)(m+1)=0$

$\therefore m_1=-2$ & $m_2=-1$ (roots are real & unequal)

$\therefore$ The C.F. is $y = c_1e^{-2x}+c_2e^{-x}$

$\therefore$ The P.I. $= \dfrac{e^{2x}}{D^2+3D+2}$

$= \dfrac{e^{2x}}{2^2+3\times2+2}$

$= \dfrac{e^{2x}}{4+6+2} = \dfrac{e^{2x}}{12}$

$\therefore$ The general solution is

$\therefore y = C.F. + P.I.$

$\therefore y = c_2e^{-x}+c_1e^{-2x}+\dfrac{e^{2x}}{12}$ Ans

③ Solve: $(D^2+a^2)y = e^{ax}$

$\Rightarrow$ Given; $(D^2+a^2)y = e^{ax}$

$\therefore$ The A.E. is $(D=m)$

$\therefore m^2+a^2=0$

$\therefore m^2=-a^2$

$\therefore m=\pm\sqrt{-a^2}$ (roots are imaginary)

$\therefore m_1 = +\sqrt{-a^2}$

$= +\sqrt{a^2} \cdot \sqrt{-1}$

$= +ai \quad [\because \sqrt{-1} = i]$

$= 0 + i \times a \quad (A + iB)$

$\therefore A = 0$ & $B = a$

$\therefore$ The C.F. is $y = e^{0 \times x}(A\cos ax + B\sin ax)$

$= A\cos ax + B\sin ax$

$\therefore$ The P.I. is $= \dfrac{e^{ax}}{D^2+a^2}$

$= \dfrac{e^{ax}}{a^2+a^2}$

$= \dfrac{e^{ax}}{2a^2}$

$\therefore$ The general solution is

$y = C.F. + P.I.$

$= A\cos ax + B\sin ax + \dfrac{e^{ax}}{2a^2}$ Ans

Now,

When X is e^{ax} (a = constant) but $f(D) = 0$ has got 'a' as its root i.e. $f(a) = 0$,

in this case, either $f(D) = (D-a)\phi(D)$

or, $f(D) = (D-a)^2$

① Find P.I. of

ⓐ $y'' + y' - 12y = e^{3x}$ $[e^{ax}]$

$\Rightarrow \dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 12y = e^{3x}$

or, $(D^2 + D - 12)y = e^{3x}$

Now, $D^2 + D - 12 = 0$

$\therefore D^2 + 4D - 3D - 12 = 0$

$\therefore D(D+4) - 3(D+4) = 0$

$\therefore (D+4)(D-3) = 0$

$\therefore D = -4 / D = 3$

as we can see one roots is a

$\therefore$ The P.I. $= \frac{e^{3x}}{(D+4)(D-3)}$

$= \frac{1}{(3+4)(D-3)} e^{3x}$ [put: $D=3$]

$= \frac{e^{3x}}{7} \cdot \frac{1}{(D+\cancel{3}-\cancel{3})}$ [put: $D = D+3$]

$= \frac{e^{3x}}{7} \cdot \frac{1}{D}(1)$

$= \frac{e^{3x}}{7} \cdot D^{-1}(1)$

$= \frac{xe^{3x}}{7}$ Ans [$\because D^{-1} = \int \therefore \int (1)\,dx = x$]

ⓑ $(D^2-4)y = e^{2x}$

$\Rightarrow \therefore$ The P.I. $= \frac{e^{2x}}{D^2-4}$ $\because D^2-4=0$
$\therefore (D+2)(D-2)=0$
$\therefore D=-2 / D=2$

$= \frac{1}{(D+2)(D-2)} e^{2x}$

$= \frac{e^{2x}}{2+2} \cdot \frac{1}{D+2-2}$ [put: $D=2$ & $D = D+2$]

$= \frac{e^{2x}}{4} \cdot \frac{1}{D}(1)$

$= \frac{e^{2x}}{4} \cdot D^{-1}(1)$

$= \frac{xe^{2x}}{4}$ Ans

ⓒ $(D-3)(D+2)y = e^{3x}$

$\therefore$ The P.I. is $= \frac{e^{3x}}{(D-3)(D+2)}$ $\because (D-3)(D+2)=0$
$\therefore D=3 / D=-2$

$= \frac{e^{3x}}{3+2} \cdot \frac{1}{D-3}$

$= \frac{e^{3x}}{5} \cdot \frac{1}{D+\cancel{3}-\cancel{3}}(1)$

$= \frac{e^{3x}}{5} \cdot D^{-1}(1)$

$= \frac{xe^{3x}}{5}$ Ans

(2) Solve: $(D^2+D-6)y=e^{2x}$

$\Rightarrow$ Given, $(D^2+D-6)y=e^{2x}$

$\therefore$ The A.E. is

$\therefore m^2+m-6=0$

or, $m^2+3m-2m-6=0$

or, $m(m+3)-2(m+3)=0$

$\therefore (m+3)(m-2)=0$

$\therefore m_1=-3 \,/\, m_2=2$ (roots are real & unequal)

$\therefore$ The C.F. is $y = c_1e^{-3x}+c_2e^{2x}$

$\therefore$ The P.I. $= \frac{e^{2x}}{D^2+D-6}$

$= \frac{e^{2x}}{(D+3)(D-2)}$

$= \frac{e^{2x}}{2+3} \cdot \frac{1}{D+\cancel{2}-\cancel{2}}(1)$

$= \frac{e^{2x}}{5} \cdot D^{-1}(1)$

$= \frac{xe^{2x}}{5}$

$\therefore$ The general solution is

$y = C.F. + P.I. = c_1e^{-3x}+c_2e^{2x}+\frac{xe^{2x}}{5}$ Ans

③ solve: $(D^2-6D+9)y=3e^{3x}$

$\Rightarrow$ Given; $(D^2-6D+9)y=3e^{3x}$

$\therefore$ The A.E. is

$\therefore m^2-6m+9=0$

or, $m^2-2\cdot m\cdot 3+3^2=0$

or, $(m-3)^2=0$

$\therefore (m-3)(m-3)=0$

$\therefore m_1=3 / m_2=3$ (roots are real & equal)

$\therefore$ The C.F. is

$y=(c_1+xc_2)e^{3x}$

$\therefore$ The P.I. is

$=\frac{1}{(D-3)^2}\cdot 3e^{3x}$

$=3e^{3x}\cdot\frac{1}{(D+\cancel{3}-\cancel{3})^2}(1)$

$=3e^{3x}\cdot\frac{1}{D^2}(1)$

$=3e^{3x}\cdot D^{-2}(1)$

$=3e^{3x}\cdot D^{-1}(D^{-1}(1))$

$=3e^{3x}\cdot D^{-1}(x)$

$=3e^{3x}\cdot\frac{x^2}{2}$

$=\frac{3}{2}x^2e^{3x}$

$\therefore$ The general solution is

$y=$ C.F. + P.I. $=(c_1+c_2x)e^{3x}+\frac{3}{2}x^2e^{3x}$ Ans