PARTIAL DIFFERENTIATION

(Mathematics - 2 | 2nd Semester)

Partial derivatives of Higher Orders:

Let's $z = f(x,y)$ then $\frac{\delta z}{\delta x}$ & $\frac{\delta z}{\delta y}$ are also the function of x & y and can be further differentiated partially w.r.t x & y

• Simbolically;

(i) $\frac{\delta}{\delta x}\left(\frac{\delta z}{\delta x}\right) = \frac{\delta^2 z}{\delta x^2}$ or $\frac{\delta^2 f}{\delta x^2}$ or f_{xx}

(ii) $\frac{\delta}{\delta y}\left(\frac{\delta z}{\delta y}\right) = \frac{\delta^2 z}{\delta y^2}$ or, $\frac{\delta^2 f}{\delta y^2}$ or f_{yy}

(iii) $\frac{\delta}{\delta x}\left(\frac{\delta z}{\delta y}\right) = \frac{\delta^2 z}{\delta x\,\delta y}$ or $\frac{\delta^2 f}{\delta x\cdot\delta y}$ or f_{xy}

(iv) $\frac{\delta}{\delta y}\left(\frac{\delta z}{\delta x}\right) = \frac{\delta^2 z}{\delta y\cdot\delta x}$ or, $\frac{\delta^2 f}{\delta y\cdot\delta x}$ or f_{yx}

& It is to be noted that $\frac{\delta^2 z}{\delta x\,\delta y} = \frac{\delta^2 z}{\delta y\cdot\delta x}$

*** Numericals**

(i) If $u = \sin^{-1}\frac{y}{x}$, then $\frac{\delta u}{\delta x} = ?$ & $\frac{\delta u}{\delta y} = ?$

⇒ Given;

$\therefore u = \sin^{-1}\frac{y}{x}$

∴ Diffentiating partially w.r.t. x, we get;

$\therefore \frac{\delta u}{\delta x} = \frac{\delta}{\delta x}\sin^{-1}\left(\frac{y}{x}\right)$ [Here; all are constant accept x & apply derivatives formulas as usual)

$= \frac{1}{\sqrt{1-\left(\frac{y}{x}\right)^2}}\cdot\frac{\delta}{\delta x}\left(\frac{y}{x}\right)$

$= \frac{x}{\sqrt{x^2-y^2}}\cdot y\cdot\frac{\delta}{\delta x}\left(\frac{1}{x}\right)$

$$= \frac{xy}{\sqrt{x^2-y^2}} \times \left(-\frac{1}{x^2}\right)$$

$$= -\frac{y}{x\sqrt{x^2-y^2}}$$ Ans

Similarly;

$$\therefore \frac{\delta u}{\delta y} = \frac{\delta}{\delta y} \sin^{-1}\left(\frac{y}{x}\right)$$

$$= \frac{1}{\sqrt{1-\left(\frac{y}{x}\right)^2}} \cdot \frac{\delta}{\delta y}\left(\frac{y}{x}\right)$$

$$= \frac{x}{\sqrt{x^2-y^2}} \cdot \frac{1}{\cancel{x}} \frac{\delta}{\delta y}(y)$$

$$= \frac{1}{\sqrt{x^2-y^2}}$$ Ans

(2) If $u = \tan^{-1}\left(\frac{y}{x}\right)$ then $\frac{\delta u}{\delta x} = ?$

$\Rightarrow$ Given; $u = \tan^{-1}\left(\frac{y}{x}\right)$

$$\therefore \frac{\delta u}{\delta x} = \frac{\delta}{\delta x} \tan^{-1}\left(\frac{y}{x}\right)$$

$$= \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{\delta}{\delta x}\left(\frac{y}{x}\right)$$

$$= \frac{x^2}{x^2+y^2} \cdot y \cdot \frac{\delta}{\delta x}\left(\frac{1}{x}\right)$$

$$= \frac{x^2}{x^2+y^2} \cdot y \cdot \left(-\frac{1}{x^2}\right)$$

$$= \frac{-y}{x^2+y^2}$$ Ans

③ If $z = x^2+y^2$, then $x \cdot \frac{\delta z}{\delta x} + y \cdot \frac{\delta z}{\delta y} = ?$

$\Rightarrow \because z = x^2+y^2$

$\therefore \frac{\delta z}{\delta x} = \frac{\delta}{\delta x}x^2 + \frac{\delta}{\delta x}y^2$

$= 2x + 0$

$= 2x$

$\therefore \frac{\delta z}{\delta y} = \frac{\delta}{\delta y}x^2 + \frac{\delta}{\delta y}y^2$

$= 0 + 2y$

$= 2y$

$\therefore x \cdot \frac{\delta z}{\delta x} + y \cdot \frac{\delta z}{\delta y}$

$= x \cdot 2x + y \cdot 2y$

$= 2x^2 + 2y^2$

$= 2(x^2+y^2)$ Ans

or,

$= 2z$ Ans

④ If $z = \frac{x}{y}$ then $x \cdot \frac{\delta z}{\delta x} + y \cdot \frac{\delta z}{\delta y} = ?$

$\Rightarrow \because z = \frac{x}{y}$

$\therefore \frac{\delta z}{\delta x} = \frac{\delta}{\delta x}\left(\frac{x}{y}\right)$

$= \frac{1}{y} \cdot \frac{\delta}{\delta x} x$

$= \frac{1}{y} \cdot 1 = \frac{1}{y}$

$\therefore \frac{\delta z}{\delta y} = \frac{\delta}{\delta y}\left(\frac{x}{y}\right)$

$= x \cdot \frac{\delta}{\delta y}\left(\frac{1}{y}\right)$

$= x \cdot \left(-\frac{1}{y^2}\right)$

$= -\frac{x}{y^2}$

$\therefore x \cdot \frac{\delta z}{\delta x} + y \cdot \frac{\delta z}{\delta y}$

$= x \times \frac{1}{y} + y \cdot \left(-\frac{x}{y^2}\right)$

$= \frac{x}{y} - \frac{x}{y}$

$= 0$ Ans

⑤ If $u = \log(x^2+y^2)$, show that $x \cdot \frac{\delta u}{\delta x} + y \cdot \frac{\delta u}{\delta y} = 2$

$\Rightarrow$ Given; $u = \log(x^2+y^2)$

$$\therefore \frac{\delta u}{\delta x} = \frac{\delta}{\delta x} \log(x^2+y^2)$$

$$= \frac{1}{x^2+y^2} \cdot \frac{\delta}{\delta x}(x^2+y^2)$$

$$= \frac{1}{x^2+y^2} \cdot \left[\frac{\delta}{\delta x} x^2 + \frac{\delta}{\delta x} y^2\right]$$

$$= \frac{1}{x^2+y^2}\left[2x+0\right]$$

$$= \frac{2x}{x^2+y^2}$$

$\therefore$ Similarly; $\frac{\delta u}{\delta y} = \frac{2y}{x^2+y^2}$

$$\therefore x \cdot \frac{\delta u}{\delta x} + y \cdot \frac{\delta u}{\delta y} = x \cdot \frac{2x}{x^2+y^2} + y \cdot \frac{2y}{x^2+y^2}$$

$$= \frac{2x^2}{x^2+y^2} + \frac{2y^2}{x^2+y^2}$$

$$= \frac{2x^2+2y^2}{x^2+y^2}$$

$$= \frac{2\cancel{(x^2+y^2)}}{\cancel{(x^2+y^2)}}$$

$= 2$ (proved) Ans

⑥ If $z = \tan^{-1}\frac{4}{x}$, then find $\frac{\delta z}{\delta x}$

$\Rightarrow$ Given; $z = \tan^{-1}\frac{4}{x}$

$$\therefore \frac{\delta z}{\delta x} = \frac{\delta}{\delta x}\tan^{-1}\frac{4}{x} = \frac{1}{1+\left(\frac{4}{x}\right)^2} \cdot \frac{\delta}{\delta x}\left(\frac{4}{x}\right) = \frac{x^2}{x^2+16} \cdot 4 \cdot \left(-\frac{1}{x^2}\right)$$

$$= -\frac{4}{x^2+16}$$ Ans

⑦ find $\frac{\delta f}{\delta x}$ & $\frac{\delta f}{\delta y}$ when $f(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$

$\Rightarrow \because f(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$

$\therefore \frac{\delta f}{\delta x} = \frac{\delta}{\delta x}\tan^{-1}\left(\frac{y}{x}\right)$

$= \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{\delta}{\delta x}\left(\frac{y}{x}\right)$

$= \frac{x^2}{x^2+y^2} \cdot y \cdot \frac{\delta}{\delta x}\left(\frac{1}{x}\right)$

$= \frac{x^2}{x^2+y^2} \cdot y \cdot \left(-\frac{1}{x^2}\right)$

$= -\frac{y}{x^2+y^2}$ Ans

$\therefore \frac{\delta f}{\delta y} = \frac{\delta}{\delta y}\tan^{-1}\left(\frac{y}{x}\right)$

$= \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{\delta}{\delta y}\left(\frac{y}{x}\right)$

$= \frac{x^2}{x^2+y^2} \cdot \frac{1}{x} \cdot \frac{\delta}{\delta y}(y)$

$= \frac{x}{x^2+y^2}$ Ans

⑧ If $u = \log(x^3+y^3+z^3-3xyz)$, then show that

$\frac{\delta u}{\delta x} + \frac{\delta u}{\delta y} + \frac{\delta u}{\delta z} = \frac{3}{x+y+z}$

$\Rightarrow$ Given;

$\therefore u = \log(x^3+y^3+z^3-3xyz)$

$\therefore \frac{\delta u}{\delta x} = \frac{\delta}{\delta x}\log(x^3+y^3+z^3-3xyz)$

$$= \frac{1}{x^3+y^3+z^3-3xyz} \cdot \frac{\delta}{\delta x}(x^3+y^3+z^3-3xyz)$$

$$= \frac{\frac{\delta}{\delta x}x^3 + \frac{\delta}{\delta x}y^3 + \frac{\delta}{\delta x}z^3 - \frac{\delta}{\delta x}3xyz}{x^3+y^3+z^3-3xyz}$$

$$= \frac{3x^2+0+0-3yz}{x^3+y^3+z^3-3xyz}$$

$$= \frac{3x^2-3yz}{x^3+y^3+z^3-3xyz}$$

$\therefore$ Similarly

$$\therefore \frac{\delta u}{\delta y} = \frac{3y^2-3xz}{x^3+y^3+z^3-3xyz} \quad \& \quad \frac{\delta u}{\delta z} = \frac{3z^3-3xy}{x^3+y^3+z^3-3xyz}$$

$\therefore$ L.H.S.

$$= \frac{\delta u}{\delta x} + \frac{\delta u}{\delta y} + \frac{\delta u}{\delta z}$$

$$= \frac{3x^2-3yz}{x^3+y^3+z^3-3xyz} + \frac{3y^2-3xz}{x^3+y^3+z^3-3xyz} + \frac{3z^3-3xy}{x^3+y^3+z^3-3xyz}$$

$$= \frac{3(x^2+y^2+z^2-xy-yz-xz)}{x^3+y^3+z^3-3xyz}$$

$$= \frac{3(x^2+y^2+z^2-xy-yz-xz)}{(x+y+z)(x^2+y^2+z^2-xy-yz-xz)}$$

$$= \frac{3}{x+y+z}$$ = R.H.S. (proved)

(9) If $u = \log(x^3+y^3+z^3-3xyz)$ then show that

$$\left(\frac{\delta}{\delta x}+\frac{\delta}{\delta y}+\frac{\delta}{\delta z}\right)^2 u = -\frac{9}{(x+y+z)^2}$$

=> Given: $u = \log(x^3+y^3+z^3-3xyz)$

$$\therefore \frac{\delta u}{\delta x} = \frac{\delta}{\delta x}\log(x^3+y^3+z^3-3xyz)$$

* By using the same technique as the previous one –

$$\therefore \frac{\delta u}{\delta x} = \frac{3x^2-3yz}{x^3+y^3+z^3-3xyz}$$

$$\therefore \frac{\delta u}{\delta y} = \frac{3y^2-3xz}{x^3+y^3+z^3-3xyz}$$

$$\therefore \frac{\delta u}{\delta z} = \frac{3z^2-3xy}{x^3+y^3+z^3-3xyz}$$

$$\therefore \frac{\delta u}{\delta x}+\frac{\delta u}{\delta y}+\frac{\delta u}{\delta z} = \frac{3(x^2+y^2+z^2-xy-yz-zx)}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)}$$

[as per the previous math)

$$\therefore \left(\frac{\delta}{\delta x}+\frac{\delta}{\delta y}+\frac{\delta}{\delta z}\right)u = \frac{3}{(x+y+z)}$$

$$\therefore \left(\frac{\delta}{\delta x}+\frac{\delta}{\delta y}+\frac{\delta}{\delta z}\right)^2 u = \left(\frac{\delta}{\delta x}+\frac{\delta}{\delta y}+\frac{\delta}{\delta z}\right)\frac{3}{(x+y+z)}$$

$$= \frac{\delta}{\delta x}\left(\frac{3}{x+y+z}\right)+\frac{\delta}{\delta y}\left(\frac{3}{x+y+z}\right)+\frac{\delta}{\delta z}\left(\frac{3}{x+y+z}\right)$$

$$= \frac{-3}{(x+y+z)^2}+\frac{-3}{(x+y+z)^2}+\frac{-3}{(x+y+z)^2}$$

$$= -\frac{9}{(x+y+z)^2}$$

= R.H.S. (proved)

(10) If $z = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$, then show that

$$x \cdot \frac{\delta z}{\delta x} + y \cdot \frac{\delta z}{\delta y} = 0$$

$\Rightarrow$ Given, $z = \tan^{-1}\left(\frac{y}{x}\right) + \sin^{-1}\left(\frac{x}{y}\right)$

$$\therefore \frac{\delta z}{\delta x} = \frac{\delta}{\delta x}\sin^{-1}\left(\frac{x}{y}\right) + \frac{\delta}{\delta x}\tan^{-1}\left(\frac{y}{x}\right)$$

$$= \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}} \cdot \frac{\delta}{\delta x}\left(\frac{x}{y}\right) + \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{\delta}{\delta x}\left(\frac{y}{x}\right)$$

$$= \frac{y}{\sqrt{y^2-x^2}} \cdot \frac{1}{y} \cdot 1 + \frac{\cancel{x^2}}{x^2+y^2} \cdot y \cdot \left(-\frac{1}{\cancel{x^2}}\right)$$

$$= \frac{1}{\sqrt{y^2-x^2}} - \frac{y}{x^2+y^2}$$

$$\therefore \frac{\delta z}{\delta y} = \frac{\delta}{\delta y}\sin^{-1}\left(\frac{x}{y}\right) + \frac{\delta}{\delta y}\tan^{-1}\left(\frac{y}{x}\right)$$

$$= \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}} \cdot \frac{\delta}{\delta y}\left(\frac{x}{y}\right) + \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{\delta}{\delta y}\left(\frac{y}{x}\right)$$

$$= \frac{y}{\sqrt{y^2-x^2}} \cdot x \cdot \frac{\delta}{\delta y}\left(\frac{1}{y}\right) + \frac{x^2}{x^2+y^2} \cdot \frac{1}{x} \cdot \frac{\delta}{\delta y}(y)$$

$$= \frac{y}{\sqrt{y^2-x^2}} \cdot x \cdot \left(-\frac{1}{y^2}\right) + \frac{x^2}{x^2+y^2} \cdot \frac{1}{x} \cdot (1)$$

$$= -\frac{x}{y\sqrt{y^2-x^2}} + \frac{x}{x^2+y^2}$$

$$\therefore x \cdot \frac{\delta z}{\delta x} = \frac{x}{\sqrt{y^2-x^2}} - \frac{xy}{x^2+y^2}$$

$$\therefore y \cdot \frac{\delta z}{\delta y} = -\frac{x}{\sqrt{y^2-x^2}} + \frac{xy}{x^2+y^2}$$

$\therefore$ L.H.S.

$= x \cdot \frac{\delta z}{\delta x} + y \cdot \frac{\delta z}{\delta y}$

$= \frac{x}{\cancel{\sqrt{y^2 - x^2}}} - \frac{xy}{\cancel{x^2 + y^2}} - \frac{x}{\cancel{\sqrt{y^2 - x^2}}} + \frac{xy}{\cancel{x^2 + y^2}}$

$= 0 =$ R.H.S. (proved)

(11) If $u = e^{xy}$, then find $\frac{\delta u}{\delta x} + \frac{\delta u}{\delta y} = ?$

$\Rightarrow$ Given, $u = e^{xy}$

$\therefore \frac{\delta u}{\delta x} = \frac{\delta}{\delta x} e^{xy}$

$= e^{xy} \cdot \frac{\delta}{\delta x}(xy)$

$= e^{xy} \cdot y \cdot \frac{\delta}{\delta x}(x)$

$= ye^{xy}$

$\therefore \frac{\delta u}{\delta y} = \frac{\delta}{\delta y} e^{xy}$

$= e^{xy} \cdot \frac{\delta}{\delta y}(xy)$

$= e^{xy} \cdot x \cdot \frac{\delta}{\delta y}(y)$

$= x \cdot e^{xy}$

$\therefore \frac{\delta u}{\delta x} + \frac{\delta u}{\delta y}$

$= ye^{xy} + xe^{xy}$

$= e^{xy}(x+y)$

$= u(x+y)$ Ans

(12) If $u = x^3 + 3x^2y + 3xy^2 + y^3$, then find $x \cdot \frac{\delta u}{\delta x} + y \cdot \frac{\delta u}{\delta y} = ?$

$\Rightarrow$ Given, $u = x^3 + 3x^2y + 3xy^2 + y^3$

$\therefore \frac{\delta u}{\delta x} = \frac{\delta}{\delta x} x^3 + \frac{\delta}{\delta x} 3x^2y + \frac{\delta}{\delta x} 3xy^2 + \frac{\delta}{\delta x} y^3$

$= 3x^2 + 3y \cdot 2x + 3y^2 \cdot 1 + 0$

$= 3x^2 + 3y^2 + 6xy$

$$\therefore \frac{\delta u}{\delta y} = \frac{\delta}{\delta y} x^3 + \frac{\delta}{\delta y} 3x^2y + \frac{\delta}{\delta y} 3xy^2 + \frac{\delta}{\delta y} y^3$$

$$= 0 + 3x^2 \cdot 1 + 3x \cdot 2y + 3y^2$$

$$= 3x^2 + 3y^2 + 6xy$$

$$\therefore x \cdot \frac{\delta u}{\delta x} + y \cdot \frac{\delta u}{\delta y}$$

$$= x(3x^2 + 3y^2 + 6xy) + y(3x^2 + 3y^2 + 6xy)$$

$$= 3x^3 + 3xy^2 + 6x^2y + 3x^2y + 3y^3 + 6xy^2$$

$$= 3x^3 + 9xy^2 + 9x^2y + 3y^3$$

$$= 3(x^3 + 3xy^2 + 3x^2y + y^3)$$

$$= 3 \times u$$

$$= 3u \text{ Ans}$$

(13) If $u = \frac{y}{z} + \frac{z}{x} + \frac{x}{y}$ then show that $x \cdot \frac{\delta u}{\delta x} + y \cdot \frac{\delta u}{\delta y} + z \cdot \frac{\delta u}{\delta z} = 0$

$$\Rightarrow \because u = \frac{y}{z} + \frac{z}{x} + \frac{x}{z}$$

$$\therefore \frac{\delta u}{\delta x} = \frac{\delta}{\delta x}\left(y/z + z/x + x/y\right)$$

$$= \frac{\delta}{\delta x}\left(\frac{y}{z}\right) + \frac{\delta}{\delta x}\left(\frac{z}{x}\right) + \frac{\delta}{\delta x}\left(\frac{x}{y}\right)$$

$$= 0 + z \cdot \left(-\frac{1}{x^2}\right) + \frac{1}{y}(1)$$

$$= \frac{1}{y} - \frac{z}{x^2}$$

$$\therefore x \cdot \frac{\delta u}{\delta x} = \frac{x}{y} - \frac{z}{x}$$

$$\therefore \frac{\delta u}{\delta y} = \frac{\delta}{\delta y}\left(\frac{y}{z}\right) + \frac{\delta}{\delta y}\left(\frac{z}{x}\right) + \frac{\delta}{\delta y}\left(\frac{x}{y}\right)$$

$$= \frac{1}{z}(1) + 0 + x \cdot \left(-\frac{1}{y^2}\right)$$

$$= \frac{1}{z} - \frac{x}{y^2}$$

$$\therefore y \cdot \frac{\delta u}{\delta y} = \frac{y}{z} - \frac{x}{y}$$

$$\therefore \frac{\delta u}{\delta z} = \frac{\delta}{\delta z}\left(\frac{y}{z}\right) + \frac{\delta}{\delta z}\left(\frac{z}{x}\right) + \frac{\delta}{\delta z}\left(\frac{x}{y}\right)$$

$$= y \cdot \left(-\frac{1}{z^2}\right) + \frac{1}{x} \cdot (1) + 0$$

$$= \frac{1}{x} - \frac{y}{z^2}$$

$$\therefore z \cdot \frac{\delta u}{\delta z} = \frac{z}{x} - \frac{y}{z}$$

L.H.S.

$$\therefore x \cdot \frac{\delta u}{\delta x} + y \cdot \frac{\delta u}{\delta y} + z \cdot \frac{\delta u}{\delta z}$$

$$= \cancel{\frac{x}{y}} - \cancel{\frac{z}{x}} + \cancel{\frac{y}{z}} - \cancel{\frac{x}{y}} + \cancel{\frac{z}{x}} - \cancel{\frac{y}{z}}$$

$= 0 =$ R.H.S. Ans

(14) If $\tan u = \frac{y}{x}$, the find the value of $\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = ?$

$$\Rightarrow \because \tan u = \frac{y}{x}$$

$$\therefore u = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\therefore \frac{\delta u}{\delta x} = \frac{\delta}{\delta x} \tan^{-1}\left(\frac{y}{x}\right)$$

$$= \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{\delta}{\delta x}\left(\frac{y}{x}\right)$$

$$= \frac{x^2}{x^2+y^2} \cdot y \cdot \left(-\frac{1}{x^2}\right)$$

$$= -\frac{y}{x^2+y^2}$$

$$\therefore \frac{\delta}{\delta x}\left(\frac{\delta u}{\delta x}\right) = \frac{\delta^2 u}{\delta x^2} = \frac{\delta}{\delta x}\left(-\frac{y}{x^2+y^2}\right)$$

$$= -y \cdot \frac{\delta}{\delta x}\left(\frac{1}{x^2+y^2}\right)$$

$$= -y \cdot \frac{(x^2+y^2) \cdot \frac{\delta}{\delta x}(1) - 1 \cdot \frac{\delta}{\delta x}(x^2+y^2)}{(x^2+y^2)^2}$$

$$= -y \cdot \frac{0-2x}{(x^2+y^2)^2}$$

$$= \frac{2xy}{(x^2+y^2)^2}$$

\# similarly

$$\frac{\delta^2 u}{\delta y^2} = \frac{-2xy}{(x^2+y^2)^2}$$

$$\therefore \frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = \frac{2xy}{(x^2+y^2)^2} - \frac{2xy}{(x^2+y^2)^2}$$

$$= 0 \text{ Ans}$$

(15) If $u = e^{xyz}$, show that $\frac{\delta^3 u}{\delta x \cdot \delta y \cdot \delta z} = e^{xyz} \cdot (1+3xyz + x^2y^2z^2)$

$\Rightarrow$ Given; $e^{xyz} = u$

$$\therefore \frac{\delta u}{\delta z} = \frac{\delta}{\delta z} e^{xyz}$$

$$= e^{xyz} \cdot \frac{\delta}{\delta z}(xyz)$$

$$= xy \cdot e^{xyz} \cdot \frac{\delta}{\delta z}(z)$$

$$= xy \cdot e^{xyz}$$

$$\therefore \frac{\delta}{\delta y}\left(\frac{\delta u}{\delta z}\right) = \frac{\delta}{\delta y}(xye^{xyz})$$

$$\therefore \frac{\delta^2 u}{\delta y \cdot \delta z} = e^{xyz} \cdot \frac{\delta}{\delta y}(xy) + xy \cdot \frac{\delta}{\delta y} e^{xyz}$$

$$= e^{xyz} \cdot x \cdot (1) + xy \cdot e^{xyz} \cdot xz \cdot (1)$$

$$= xe^{xyz} + x^2yz \cdot e^{xyz}$$

$$= e^{xyz}(x + x^2yz)$$

$$\therefore \frac{\delta}{\delta x}\left(\frac{\delta^2 u}{\delta y \cdot \delta z}\right) = \frac{\delta}{\delta x}\left[e^{xyz}(x + x^2yz)\right]$$

$$= e^{xyz} \cdot \frac{\delta}{\delta x}(x + x^2yz) + (x + x^2yz) \cdot \frac{\delta}{\delta x} e^{xyz}$$

$$= e^{xyz} \cdot (1 + yz \cdot 2x) + (x + x^2yz) \cdot e^{xyz} \cdot yz \cdot (1)$$

$$= e^{xyz}(1 + 2xyz + xyz + x^2y^2z^2)$$

$$= e^{xyz}(1 + 3xyz + x^2y^2z^2)$$

= R.H.S. (proved)