WBSCTE — Diploma / Polytechnic, 1st Semester

Mathematics I

Chapter: BINOMIAL THEOREM

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1. Introduction

The Binomial Theorem gives a formula for expanding any power of a binomial expression, i.e. an expression of the form $(x + a)^n$, without multiplying it out term by term. It is one of the most frequently tested chapters of Mathematics I because it combines simple algebraic manipulation with the concept of combinations (${}^{n}C_r$). As per the WBSCTE syllabus, this chapter covers the Binomial Theorem for positive integral index (statement only, no proof needed), simple problems on positive index, and expansion of $(1+x)^{-1}$, $(1-x)^{-1}$ and similar series for any index where $|x| < 1$.

2. Statement of the Binomial Theorem (Positive Integral Index)

For any positive integer n, and real numbers x, a:

- $(x+a)^n = {}^{n}C_0\, x^n + {}^{n}C_1\, x^{n-1}a + {}^{n}C_2\, x^{n-2}a^2 + \ldots + {}^{n}C_r\, x^{n-r}a^r + \ldots + {}^{n}C_n\, a^n$

Here, ${}^{n}C_r = n! / [r!(n-r)!]$ is called the binomial coefficient. The expansion always has exactly (n+1) terms.

3. General Term and Middle Term

- **General term:** $T_{r+1} = {}^{n}C_r\, x^{n-r} a^r$
- **If n is EVEN: one middle term =** $T_{(n/2+1)}$
- **If n is ODD: two middle terms =** $T_{(n+1)/2}$ **and** $T_{(n+3)/2}$

4. Important Properties of Binomial Coefficients

- ${}^{n}C_0 = {}^{n}C_n = 1$
- ${}^{n}C_r = {}^{n}C_{n-r}$
- ${}^{n}C_r + {}^{n}C_{r-1} = {}^{n+1}C_r$ **(Pascal's Rule)**
- ${}^{n}C_0 + {}^{n}C_1 + {}^{n}C_2 + \ldots + {}^{n}C_n = 2^n$
- ${}^{n}C_0 - {}^{n}C_1 + {}^{n}C_2 - \ldots + (-1)^n\, {}^{n}C_n = 0 \quad (n \geq 1)$

5. Binomial Theorem for Any Index (Negative / Rational), |x| < 1

When n is negative or a fraction, the expansion of $(1+x)^n$ does NOT terminate — it becomes an infinite series, valid only when $|x| < 1$:

- $(1+x)^n = 1 + nx + [n(n-1)/2!]x^2 + [n(n-1)(n-2)/3!]x^3 + \ldots \quad (|x| < 1)$

Standard expansions to memorise (very frequently used directly in problems):

- $(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots$

- $(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots$
- $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \dots$
- $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots$
- $(1+x)^{-3} = 1 - 3x + 6x^2 - 10x^3 + \dots$
- $(1-x)^{-3} = 1 + 3x + 6x^2 + 10x^3 + \dots$

6. Important Points for Exams

- Total number of terms in $(x+a)^n$ is always $(n+1)$, never n.
- Always identify r correctly: for the kth term, use $r = k - 1$ in the general term formula T_{r+1}.
- For 'term independent of x' problems, set the total power of x in the general term equal to zero and solve for r.
- For negative/fractional index expansions, the series never terminates and is valid only for $|x| < 1$ — this condition is often asked to be stated.
- Approximation-type questions (e.g., evaluate $(101)^4$ or $\sqrt{99}$) are solved by writing the number as (a ± b) with a small b, then applying the expansion and taking only the first few terms.

7. 30 Important Questions & Answers (With Step-by-Step Solutions)

The following questions are compiled based on the WBSCTE Mathematics I syllabus for the Binomial Theorem chapter and repeatedly tested question patterns seen in external examination papers. Practising all of these thoroughly gives very strong exam coverage.

Q1. State the Binomial Theorem for a positive integral index n (statement only).

[Core statement — asked almost every year]

Solution:

For any positive integer n, and real numbers x, a:

$(x+a)^n = {}^{nC}{}_0\, x^n + {}^{nC}{}_1\, x^{n-1}a + {}^{nC}{}_2\, x^{n-2}a^2 + \ldots + {}^{nC}{}_r\, x^{n-r}a^r + \ldots + {}^{nC}{}_n\, a^n$

where ${}^{nC}{}_r = n! / [r!(n-r)!]$ is the binomial coefficient of the (r+1)th term.

The expansion has exactly (n+1) terms.

Answer: $(x+a)^n = \Sigma\ {}^{nC}{}_r\, x^{n-r}\, a^r$, r = 0 to n (n+1 terms in total)

Q2. Expand $(x + y)^4$ using the Binomial Theorem.

[Direct expansion — very common]

Solution:

Using $(x+y)^4 = {}^{4C}{}_0\, x^4 + {}^{4C}{}_1\, x^{3y} + {}^{4C}{}_2\, x^{2y2} + {}^{4C}{}_3\, xy^3 + {}^{4C}{}_4\, y^4$.

${}^{4C}{}_0=1$, ${}^{4C}{}_1=4$, ${}^{4C}{}_2=6$, ${}^{4C}{}_3=4$, ${}^{4C}{}_4=1$.

So $(x+y)^4 = x^4 + 4x^{3y} + 6x^{2y2} + 4xy^3 + y^4$.

Answer: $(x+y)^4 = x^4 + 4x^{3y} + 6x^{2y2} + 4xy^3 + y^4$

Q3. Expand $(1 - x)^5$ using the Binomial Theorem.

[Direct expansion with alternating sign — very common]

Solution:

$(1-x)^5 = {}^{5C}{}_0 - {}^{5C}{}_1\, x + {}^{5C}{}_2\, x^2 - {}^{5C}{}_3\, x^3 + {}^{5C}{}_4\, x^4 - {}^{5C}{}_5\, x^5$.

${}^{5C}{}_0=1$, ${}^{5C}{}_1=5$, ${}^{5C}{}_2=10$, ${}^{5C}{}_3=10$, ${}^{5C}{}_4=5$, ${}^{5C}{}_5=1$.

So $(1-x)^5 = 1 - 5x + 10x^2 - 10x^3 + 5x^4 - x^5$.

Answer: $(1-x)^5 = 1 - 5x + 10x^2 - 10x^3 + 5x^4 - x^5$

Q4. Expand $(2x - 3y)^4$ using the Binomial Theorem.

[Expansion with numerical coefficients — common exam pattern]

Solution:

$(2x-3y)^4 = {}^{4C}{}_0(2x)^4 - {}^{4C}{}_1(2x)^3(3y) + {}^{4C}{}_2(2x)^2(3y)^2 - {}^{4C}{}_3(2x)(3y)^3 + {}^{4C}{}_4(3y)^4$.

$T1 = 16x^4$. $T2 = -4\cdot 8x^3\cdot 3y = -96x^{3y}$. $T3 = 6\cdot 4x^2\cdot 9y^2 = 216x^{2y2}$.

$T4 = -4\cdot 2x\cdot 27y^3 = -216xy^3$. $T5 = 81y^4$.

So $(2x-3y)^4 = 16x^4 - 96x^{3y} + 216x^{2y2} - 216xy^3 + 81y^4$.

Answer: $16x^4 - 96x^{3y} + 216x^{2y2} - 216xy^3 + 81y^4$

Q5. Find the general term in the expansion of $(x + a)^n$. Hence find the 5th term of $(2x + 3)^7$.

[General-term formula application — very important]

Solution:

General term: $T_{r+1} = {}^{nC}{}_r\, x^{n-r}\, a^r$.

For the 5th term, r+1=5 ⟹=4. Here n=7, x→2x, a→3.

$T5 = {}^{7C}{}_4\, (2x)^{7-4}\, (3)^4 = 35 \cdot (2x)^3 \cdot 81$.

$= 35 \cdot 8x^3 \cdot 81 = 35 \cdot 648\, x^3 = 22680x^3$.

Answer: $T5 = 22680x^3$

Q6. Find the 4th term in the expansion of $(x + 2)^8$.

[Direct general-term application]

Solution:

$T_{r+1} = {}^{8C}{}_r\, x^{8-r}\, 2^r$. For the 4th term, r+1=4 ⟹=3.

$T4 = {}^{8C}{}_3\, x^{8-3}\, 2^3 = 56 \cdot x^5 \cdot 8$.

$= 448x^5$.

Answer: $T4 = 448x^5$

Q7. Find the middle term in the expansion of $(x + 1/x)^{10}$.

[Middle-term type — even index — very common]

Solution:

Here n=10 (even), so there is exactly one middle term: $T_{(n/2}+1) = T6$.

T6: r+1=6 ⟹=5. $T6 = {}^{10C}{}_5\, x^{10-5}\, (1/x)^5 = 252 \cdot x^5 \cdot x^{-5}$.

$= 252 \cdot x^0 = 252$.

Answer: Middle term = 252 (a constant term)

Q8. Find the middle term(s) in the expansion of $(2x - y)^7$.

[Middle-term type — odd index — very common]

Solution:

Here n=7 (odd), so there are two middle terms: T4 and T5 [positions (n+1)/2 and (n+3)/2].

T4: r=3. $T4 = {}^{7C}{}_3\, (2x)^4(-y)^3 = 35 \cdot 16x^4 \cdot (-y^3) = -560x^{4y3}$.

T5: r=4. $T5 = {}^{7C}{}_4\, (2x)^3(-y)^4 = 35 \cdot 8x^3 \cdot y^4 = 280x^{3y4}$.

Answer: Middle terms: $-560x^{4y3}$ and $280x^{3y4}$

Q9. Find the coefficient of x^5 in the expansion of $(1 + x)^9$.

[Basic coefficient-finding question]

Solution:

General term: $T_{r+1} = {}^{9}C_r\, x^r$. Coefficient of x^5 means $r=5$.

Coefficient $= {}^{9}C_5 = 9!/(5!4!) = 126$.

Answer: Coefficient of x^5 = 126

Q10. Find the term independent of x in the expansion of $(2x^2 - 1/x)^{12}$.

[Important application-type question]

Solution:

$T_{r+1} = {}^{12}C_r\, (2x^2)^{12-r} (-1/x)^r = {}^{12}C_r\, 2^{12-r} (-1)^r x^{24-2r-r} = {}^{12}C_r\, 2^{12-r}(-1)^r x^{24-3r}$.

For the term independent of x, set the power of x to zero: $24 - 3r = 0 \Rightarrow r = 8$.

Term $= {}^{12}C_8 \cdot 2^{12-8} \cdot (-1)^8 = {}^{12}C_8 \cdot 16 \cdot 1$.

${}^{12}C_8 = {}^{12}C_4 = 495$, so the term $= 495 \times 16 = 7920$.

Answer: Term independent of x = 7920

Q11. Using the Binomial Theorem, prove that ${}^{n}C_0 + {}^{n}C_1 + {}^{n}C_2 + \dots + {}^{n}C_n = 2^n$.

[Standard proof — asked frequently]

Solution:

Start from $(1+x)^n = {}^{n}C_0 + {}^{n}C_1\, x + {}^{n}C_2\, x^2 + \dots + {}^{n}C_n\, x^n$.

Put x = 1 on both sides: $(1+1)^n = {}^{n}C_0 + {}^{n}C_1 + {}^{n}C_2 + \dots + {}^{n}C_n$.

So $2^n = {}^{n}C_0 + {}^{n}C_1 + \dots + {}^{n}C_n$. (Proved)

Answer: ${}^{n}C_0 + {}^{n}C_1 + \dots + {}^{n}C_n = 2^n$ (proved)

Q12. Using the Binomial Theorem, prove that ${}^{n}C_0 - {}^{n}C_1 + {}^{n}C_2 - \dots + (-1)^n\, {}^{n}C_n = 0$, for $n \geq 1$.

[Standard proof — asked frequently]

Solution:

Start from $(1+x)^n = {}^{n}C_0 + {}^{n}C_1\, x + {}^{n}C_2\, x^2 + \dots + {}^{n}C_n\, x^n$.

Put $x = -1$: $(1-1)^n = {}^{n}C_0 - {}^{n}C_1 + {}^{n}C_2 - \dots + (-1)^n\, {}^{n}C_n$.

Since $n \geq 1$, LHS $= 0^n = 0$.

So ${}^{n}C_0 - {}^{n}C_1 + {}^{n}C_2 - \dots + (-1)^n\, {}^{n}C_n = 0$. (Proved)

Answer: Alternating sum of binomial coefficients = 0 (proved)

Q13. Prove that ${}^{n}C_r = {}^{n}C_{n-r}$.

[Standard identity proof]

Solution:

By definition, ${}^{n}C_r = n! / [r!(n-r)!]$.

Now ${}^{n}C_{n-r}$ = n! / [(n−r)! (n−(n−r))!] = n! / [(n−r)! r!].

This is identical to ${}^{n}C_{r}$.

Hence ${}^{n}C_{r} = {}^{n}C_{n-r}$. (Proved)

Answer: ${}^{n}C_{r} = {}^{n}C_{n-r}$ (proved)

Q14. Prove Pascal's Rule: ${}^{n}C_{r} + {}^{n}C_{r-1} = {}^{n+1}C_{r}$.

[Standard identity proof — important]

Solution:

${}^{n}C_{r} + {}^{n}C_{r-1}$ = n!/[r!(n−r)!] + n!/[(r−1)!(n−r+1)!].

Take n!/[(r−1)!(n−r)!] common: = n!/[(r−1)!(n−r)!] × [1/r + 1/(n−r+1)].

= n!/[(r−1)!(n−r)!] × [(n−r+1+r) / (r(n−r+1))] = n!/[(r−1)!(n−r)!] × [(n+1)/(r(n−r+1))].

= (n+1)! / [r!(n−r+1)!] = ${}^{n+1}C_{r}$.

Hence ${}^{n}C_{r} + {}^{n}C_{r-1} = {}^{n+1}C_{r}$. (Proved)

Answer: ${}^{n}C_{r} + {}^{n}C_{r-1} = {}^{n+1}C_{r}$ (proved)

Q15. Using Pascal's Rule, find the value of ${}^{10}C_{4} + {}^{10}C_{3}$.

[Direct application of Pascal's Rule]

Solution:

By Pascal's Rule, ${}^{10}C_{4} + {}^{10}C_{3} = {}^{11}C_{4}$.

${}^{11}C_{4}$ = 11!/(4!7!) = 330.

(Check: ${}^{10}C_{4}$ = 210, ${}^{10}C_{3}$ = 120, and 210+120 = 330 √)

Answer: ${}^{10}C_{4} + {}^{10}C_{3}$ = 330

Q16. Find the number of terms in the expansion of $(2x - 3y)^8$.

[Basic concept question]

Solution:

For any binomial expansion $(a+b)^n$, the number of terms is always (n+1).

Here n = 8, so the number of terms = 8 + 1 = 9.

Answer: Number of terms = 9

Q17. Using the Binomial Theorem, evaluate $(101)^4$.

[Classic application — very common in exams]

Solution:

Write 101 = 100 + 1, so $(101)^4 = (100+1)^4$.

= $100^4 + 4 \cdot 100^3 \cdot 1 + 6 \cdot 100^2 \cdot 1^2 + 4 \cdot 100 \cdot 1^3 + 1^4$.

= 100000000 + 4000000 + 60000 + 400 + 1.

$= 104060401$.

Answer: $(101)^4 = 104060401$

Q18. Using the Binomial Theorem, evaluate $(0.98)^4$ correct to four decimal places.

[Classic application — approximation type]

Solution:

Write $0.98 = 1 - 0.02$, so $(0.98)^4 = (1-0.02)^4$.

$= 1 - 4(0.02) + 6(0.02)^2 - 4(0.02)^3 + (0.02)^4$.

$= 1 - 0.08 + 0.0024 - 0.000032 + 0.00000016$.

$= 0.92236816 \approx 0.9224$ (to 4 decimal places).

Answer: $(0.98)^4 \approx 0.9224$

Q19. Find the sum of the coefficients in the expansion of $(3x - 4y)^7$.

[Standard shortcut question — set x=y=1]

Solution:

The sum of all coefficients in an expansion of $f(x,y)$ is obtained by putting $x = y = 1$.

Sum $= (3\cdot1 - 4\cdot1)^7 = (3-4)^7 = (-1)^7$.

$= -1$.

Answer: Sum of coefficients = −1

Q20. Expand $(1 + x)^{-1}$ up to 4 terms ($|x| < 1$).

[Standard negative-index series — very important]

Solution:

Using $(1+x)^n = 1 + nx + [n(n-1)/2!]x^2 + [n(n-1)(n-2)/3!]x^3 + \ldots$ with $n = -1$:

Term2: $(-1)x = -x$. Term3: $[(-1)(-2)/2]x^2 = x^2$. Term4: $[(-1)(-2)(-3)/6]x^3 = -x^3$.

So $(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots$

Answer: $(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots$

Q21. Expand $(1 - x)^{-2}$ up to 4 terms ($|x| < 1$).

[Standard negative-index series — very important]

Solution:

Using $(1+u)^n$ with $n = -2$, $u = -x$: $(1-x)^{-2} = 1 + (-2)(-x) + [(-2)(-3)/2!](-x)^2 + [(-2)(-3)(-4)/3!](-x)^3 + \ldots$

$= 1 + 2x + 3x^2 + 4x^3 + \ldots$

(General coefficient pattern for $(1-x)^{-2}$ is simply $(r+1)$ for the term in x^r.)

Answer: $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots$

Q22. Expand $(1 + x)^{1/2}$ up to the first four terms ($|x| < 1$).

[Standard fractional-index series — important]

Solution:

Using $(1+x)^n = 1 + nx + [n(n-1)/2!]x^2 + [n(n-1)(n-2)/3!]x^3 + \ldots$ with $n = 1/2$:

Term2: $(1/2)x$.

Term3: $[(1/2)(-1/2)/2]x^2 = (-1/4 \div 2)x^2 = -x^2/8$.

Term4: $[(1/2)(-1/2)(-3/2)/6]x^3 = (3/8 \div 6)x^3 = x^3/16$.

So $(1+x)^{1/2} \approx 1 + x/2 - x^2/8 + x^3/16 - \ldots$

Answer: $(1+x)^{1/2} \approx 1 + x/2 - x^2/8 + x^3/16 - \ldots$

Q23. Using the Binomial Theorem, find the approximate value of √99 correct to four decimal places.

[Classic application — square-root approximation]

Solution:

Write $99 = 100(1 - 0.01)$, so $\sqrt{99} = 10(1 - 0.01)^{1/2}$.

$(1-0.01)^{1/2} \approx 1 - (1/2)(0.01) - (1/8)(0.01)^2 = 1 - 0.005 - 0.0000125 = 0.9949875$.

So $\sqrt{99} \approx 10 \times 0.9949875 = 9.949875 \approx 9.9499$.

Answer: $\sqrt{99} \approx 9.9499$

Q24. Find the coefficient of x^n in the expansion of $(1 - x)^{-2}$, and hence write the general term.

[Standard negative-index general-term question]

Solution:

For $(1-x)^{-m}$, the general term is $T_{r+1} = {}^{m+r-1}C_r\, x^r$.

For $m = 2$: $T_{r+1} = {}^{r+1}C_r\, x^r = (r+1)\, x^r$ [since ${}^{r+1}C_r = {}^{r+1}C_1 = r+1$].

So the coefficient of x^n is simply $(n+1)$.

Answer: Coefficient of x^n in $(1-x)^{-2}$ is $(n+1)$

Q25. Expand $(1 + 2x)^{-1}$ up to 4 terms ($|2x| < 1$).

[Standard negative-index series application]

Solution:

Treat $u = 2x$: $(1+u)^{-1} = 1 - u + u^2 - u^3 + \ldots$

Substitute $u = 2x$: $1 - 2x + (2x)^2 - (2x)^3 = 1 - 2x + 4x^2 - 8x^3$.

Answer: $(1+2x)^{-1} = 1 - 2x + 4x^2 - 8x^3 + \ldots$

Q26. If the coefficients of the 2nd, 3rd and 4th terms in the expansion of $(1+x)^n$ are in Arithmetic Progression (AP), find n.

[Classic important long-answer question]

Solution:

Coefficients: T2 → ${}^{n}C_1 = n$, T3 → ${}^{n}C_2 = n(n-1)/2$, T4 → ${}^{n}C_3 = n(n-1)(n-2)/6$.

AP condition: $2\cdot{}^{n}C_2 = {}^{n}C_1 + {}^{n}C_3$.

$2 \cdot n(n-1)/2 = n + n(n-1)(n-2)/6 \implies n(n-1) = n + n(n-1)(n-2)/6$.

Dividing throughout by n ($n \neq 0$): $(n-1) = 1 + (n-1)(n-2)/6$.

Multiply by 6: $6(n-1) = 6 + (n-1)(n-2) \implies 6n-6 = 6 + n^2-3n+2$.

$\implies n^2 - 9n + 14 = 0 \implies (n-7)(n-2) = 0 \implies n = 7$ or $n = 2$.

n = 2 is rejected since the 4th term would not exist for n = 2. Hence n = 7.

Answer: n = 7

Q27. Find the coefficient of x^6y^3 in the expansion of $(x + y)^9$.

[Direct coefficient question]

Solution:

General term: $T_{r+1} = {}^{9}C_r\, x^{9-r}\, y^r$. For x^{6y3}, r = 3.

Coefficient = ${}^{9}C_3 = 9!/(3!6!) = 84$.

Answer: Coefficient of x^6y^3 = 84

Q28. If the coefficients of x^7 and x^8 in the expansion of $(2 + x/3)^n$ are equal, find n.

[Classic important long-answer question]

Solution:

General term: $T_{r+1} = {}^{n}C_r\, 2^{n-r}\, (x/3)^r = {}^{n}C_r\, 2^{n-r}\, x^r / 3^r$.

Coefficient of x^7 (r=7): ${}^{n}C_7 \cdot 2^{n-7}/3^7$. Coefficient of x^8 (r=8): ${}^{n}C_8 \cdot 2^{n-8}/3^8$.

Equate: ${}^{n}C_7 \cdot 2^{n-7}/3^7 = {}^{n}C_8 \cdot 2^{n-8}/3^8 \implies 6\cdot{}^{n}C_7 = {}^{n}C_8$.

Using ${}^{n}C_8 / {}^{n}C_7 = (n-7)/8$, we get $(n-7)/8 = 6 \implies n-7 = 48$.

$\implies n = 55$.

Answer: n = 55

Q29. Find the middle term in the expansion of $(x/3 + 9y)^{10}$.

[Middle-term application with fractional/large coefficients]

Solution:

n = 10 (even), so there is one middle term: T6 (r=5).

$T6 = {}^{10}C_5\, (x/3)^5\, (9y)^5 = 252 \cdot (x^5/3^5) \cdot 9^5\, y^5$.

$9^5/3^5 = (9/3)^5 = 3^5 = 243$.

$T6 = 252 \times 243 \times x^{5y5} = 61236\, x^{5y5}$.

Answer: Middle term = $61236\, x^5y^5$

Q30. Find the general term in the expansion of $(1 + x)^{15}$ and hence find the coefficient of x^{10}.

[Wrap-up question combining general term + coefficient]

Solution:

General term: $T_{r+1} = {}^{15}C_r \, x^r$.

For the coefficient of x^{10}, put $r = 10$.

Coefficient = ${}^{15}C_{10} = {}^{15}C_5 = 15!/(5!10!) = 3003$.

Answer: Coefficient of x^{10} = 3003
