WBSCTE — Diploma / Polytechnic, 1st Semester

Mathematics I

Chapter: COMPLEX NUMBER

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1. Introduction

A Complex Number is a number of the form $z = x + iy$, where x and y are real numbers and $i = \sqrt{(-1)}$ is called iota, the imaginary unit, satisfying $i^2 = -1$. Here x is called the real part of z, written Re(z), and y is called the imaginary part, written Im(z). This chapter (Unit 1.2 of the WBSCTE Mathematics I syllabus) covers the algebra of complex numbers, the conjugate, modulus and amplitude (argument), and the polar/exponential forms of a complex number. It is a highly formula-driven, scoring chapter that also connects directly to solving quadratic equations with no real roots.

2. Powers of iota (i)

Since $i^2=-1$, the powers of i repeat in a cycle of 4:

- **$i^1 = i$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$**

To simplify i^n for any large n, divide n by 4 and use the remainder: $i^n = i^{\text{remainder}}$.

3. Algebra of Complex Numbers

If $z_1 = a+bi$ and $z_2 = c+di$:

- **Equality: $z_1 = z_2 \iff a=c$ and $b=d$**
- **Addition: $z_{1+z2} = (a+c) + (b+d)i$**
- **Subtraction: $z_1-z_2 = (a-c) + (b-d)i$**
- **Multiplication: $z_1 \cdot z_2 = (ac-bd) + (ad+bc)i$**

Division is performed by multiplying the numerator and denominator by the conjugate of the denominator, to remove i from the denominator.

4. Conjugate, Modulus and Amplitude

For $z = x+iy$:

- **Conjugate: $\bar{z} = x - iy$**
- **Modulus: $|z| = \sqrt{(x^2+y^2)}$**
- **Amplitude (Argument): $\theta = \tan^{-1}(y/x)$ (adjusted for the correct quadrant of x, y)**
- **$z \cdot \bar{z} = |z|^2$, $z + \bar{z} = 2 \cdot \text{Re}(z)$**

5. Polar (Modulus-Amplitude) Form and Exponential Form

Every complex number $z=x+iy$ can be written using its modulus $r=|z|$ and amplitude θ:

- **Polar form: $z = r(\cos\theta + i \sin\theta)$**
- **Exponential form: $z = r \cdot e^{i\theta}$ (using Euler's formula $e^{i\theta}=\cos\theta+i \sin\theta$)**

6. Important Properties for Exams

- $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$, and $|z_1/z_2| = |z_1|/|z_2|$ (for $z_2 \neq 0$)
- $\arg(z_1 \cdot z_2) = \arg(z_1) + \arg(z_2)$
- Conjugate of a sum/product = sum/product of the conjugates.
- When solving a quadratic $ax^2+bx+c=0$ with negative discriminant ($b^2-4ac<0$), the roots are complex conjugates of each other.
- Always determine the correct quadrant of (x,y) before finalising the amplitude — the reference angle from $\tan^{-1}(|y|/|x|)$ must be adjusted using the quadrant rules (Q1: θ, Q2: $180°-\theta$, Q3: $-(180°-\theta)$, Q4: $-\theta$).

7. 30 Important Questions & Answers (With Step-by-Step Solutions)

The following questions are compiled based on the WBSCTE Mathematics I syllabus for the Complex Number chapter and repeatedly tested question patterns seen in external examination papers. Practising all of these thoroughly gives very strong exam coverage.

Q1. Simplify: i^{15}.

[Basic — powers of iota — asked almost every year]

Solution:

Powers of i repeat in a cycle of 4: $i^1=i$, $i^2=-1$, $i^3=-i$, $i^4=1$.

$15 = 4\times3 + 3$, so $i^{15} = i^{4\times3+3} = (i^4)^3 \cdot i^3 = 1^3 \cdot i^3$.

$= i^3 = -i$.

Answer: $i^{15} = -i$

Q2. Simplify: i^{-1}.

[Basic — negative power of iota]

Solution:

$i^{-1} = 1/i$.

Multiply numerator and denominator by i: $1/i \times i/i = i/i^2 = i/(-1)$.

$= -i$.

Answer: $i^{-1} = -i$

Q3. If $z_1 = 3+4i$ and $z_2 = 1-2i$, find $z_1 + z_2$.

[Basic — addition of complex numbers]

Solution:

Add real parts and imaginary parts separately: $(3+1) + (4+(-2))i$.

$= 4 + 2i$.

Answer: $z_{1+z2} = 4+2i$

Q4. If $z_1 = 3+4i$ and $z_2 = 1-2i$, find $z_1 - z_2$.

[Basic — subtraction of complex numbers]

Solution:

Subtract real parts and imaginary parts separately: $(3-1) + (4-(-2))i$.

$= 2 + 6i$.

Answer: $z_1-z_2 = 2+6i$

Q5. Multiply: $(2+3i)(1-i)$.

[Basic — multiplication of complex numbers — very common]

Solution:

$(2+3i)(1-i) = 2(1) + 2(-i) + 3i(1) + 3i(-i)$.

$= 2 - 2i + 3i - 3i^2$.

Since $i^2=-1$: $= 2 + i + 3 = 5 + i$.

Answer: $5 + i$

Q6. Divide: $(3+4i)/(1-2i)$.

[Division by rationalization — very important]

Solution:

Multiply numerator and denominator by the conjugate of the denominator, $(1+2i)$:

Numerator: $(3+4i)(1+2i) = 3+6i+4i+8i^2 = 3+10i-8 = -5+10i$.

Denominator: $(1)^2+(2)^2 = 1+4 = 5$.

Result $= (-5+10i)/5 = -1+2i$.

Answer: $-1 + 2i$

Q7. Find the conjugate of $z = 5 - 7i$.

[Basic — definition of conjugate]

Solution:

The conjugate of $x+iy$ is $x-iy$ (sign of imaginary part reversed).

Conjugate of $5-7i = 5+7i$.

Answer: $\bar{z} = 5 + 7i$

Q8. Find the modulus of $z = 3 + 4i$.

[Basic — modulus formula — asked almost every year]

Solution:

$|z| = \sqrt{(x^2+y^2)}$, where $z=x+iy$.

$|z| = \sqrt{(3^2+4^2)} = \sqrt{(9+16)} = \sqrt{25}$.

$= 5$.

Answer: $|z| = 5$

Q9. Find the modulus of $z = -5 + 12i$.

[Basic — modulus formula]

Solution:

$|z| = \sqrt{((-5)^2+12^2)} = \sqrt{(25+144)} = \sqrt{169}$.

$= 13$.

Answer: |z| = 13

Q10. Find the amplitude (argument) of z = 1 + i.

[Basic — argument in 1st quadrant]

Solution:

$\tan\theta = y/x = 1/1 = 1$.

Since $x>0$ and $y>0$, z lies in the 1st quadrant, so θ is the reference angle itself.

$\theta = 45° = \pi/4$.

Answer: Amplitude = π/4 (45°)

Q11. Find the amplitude of z = −1 + i.

[Argument in 2nd quadrant — important]

Solution:

Reference angle: $\tan^{-1}(|y|/|x|) = \tan^{-1}(1/1) = 45°$.

Since $x<0$ and $y>0$, z lies in the 2nd quadrant, so $\theta = 180° - 45°$.

$= 135°$.

Answer: Amplitude = 135°

Q12. Find the amplitude of z = −1 − i.

[Argument in 3rd quadrant — important]

Solution:

Reference angle = 45°. Since $x<0$ and $y<0$, z lies in the 3rd quadrant.

Using the principal value range $(-180°, 180°]$, $\theta = -(180° - 45°)$.

$= -135°$.

Answer: Amplitude = −135°

Q13. Find the amplitude of z = 1 − i.

[Argument in 4th quadrant — important]

Solution:

Reference angle = $\tan^{-1}(1/1) = 45°$. Since $x>0$ and $y<0$, z lies in the 4th quadrant.

$\theta = -45°$.

Answer: Amplitude = −45°

Q14. Express z = 1 + i√3 in polar (modulus-amplitude) form.

[Polar form conversion — very important]

Solution:

$|z| = \sqrt{1^2+(\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2$.

$\theta = \tan^{-1}(\sqrt{3}/1) = 60°$ (z is in the 1st quadrant).

Polar form: $z = 2(\cos 60° + i \sin 60°)$.

Answer: $z = 2(\cos 60° + i \sin 60°)$

Q15. Express $z = -2 + 2i$ in polar (modulus-amplitude) form.

[Polar form conversion — important]

Solution:

$|z| = \sqrt{(-2)^2+2^2} = \sqrt{8} = 2\sqrt{2}$.

Reference angle $= \tan^{-1}(2/2) = 45°$; since $x<0$, $y>0$, z is in the 2nd quadrant, so $\theta=180°-45°=135°$.

Polar form: $z = 2\sqrt{2}\,(\cos 135° + i \sin 135°)$.

Answer: $z = 2\sqrt{2}\,(\cos 135° + i \sin 135°)$

Q16. Convert $z = 4(\cos 60° + i \sin 60°)$ into Cartesian (rectangular) form.

[Reverse polar-to-Cartesian conversion — important]

Solution:

$x = r \cos\theta = 4 \cos 60° = 4 \times (1/2) = 2$.

$y = r \sin\theta = 4 \sin 60° = 4 \times (\sqrt{3}/2) = 2\sqrt{3}$.

So $z = 2 + 2\sqrt{3}\, i$.

Answer: $z = 2 + 2\sqrt{3}\, i$

Q17. Express $z = 1 + i$ in exponential form.

[Exponential (Euler) form — important]

Solution:

$|z| = \sqrt{1^2+1^2} = \sqrt{2}$.

$\theta = \tan^{-1}(1/1) = \pi/4$ (45°).

Exponential form: $z = \sqrt{2} \cdot e^{i\pi/4}$ [using Euler's formula $z = r\cdot e^{i\theta}$].

Answer: $z = \sqrt{2}\, e^{i\pi/4}$

Q18. If $z_1=2+3i$, $z_2=1-i$, verify that $|z_1 \cdot z_2| = |z_1|\cdot|z_2|$.

[Standard property verification — important]

Solution:

$z_1z_2 = (2+3i)(1-i) = 2-2i+3i-3i^2 = 2+i+3 = 5+i$.

$|z_1z_2| = \sqrt{5^2+1^2} = \sqrt{26}$.

$|z_1| = \sqrt{2^2+3^2} = \sqrt{13}$. $|z_2| = \sqrt{1^2+1^2} = \sqrt{2}$.

$|z_1|\cdot|z_2| = \sqrt{13} \times \sqrt{2} = \sqrt{26} = |z_1z_2|$. (Verified)

Answer: $|z_1z_2| = |z_1||z_2| = \sqrt{26}$ (verified)

Q19. If $x+iy = (3+4i)/(1-2i)$, find the values of x and y.

[Equality of complex numbers — classic important question]

Solution:

Rationalize: multiply numerator and denominator by $(1+2i)$.

Numerator: $(3+4i)(1+2i) = 3+6i+4i+8i^2 = 3+10i-8 = -5+10i$.

Denominator: $1^2+2^2 = 5$.

So $x+iy = (-5+10i)/5 = -1+2i$. Comparing real and imaginary parts: $x=-1$, $y=2$.

Answer: $x = -1$, $y = 2$

Q20. If $(x-iy)(3+5i)$ is the conjugate of $(-6-24i)$, find x and y.

[Classic important long-answer question — equality of complex numbers]

Solution:

Conjugate of $(-6-24i)$ is $(-6+24i)$. So $(x-iy)(3+5i) = -6+24i$.

Expand LHS: $3x+5xi-3yi-5yi^2 = (3x+5y) + (5x-3y)i$.

Equate real parts: $3x+5y = -6$. Equate imaginary parts: $5x-3y = 24$.

Solving the two equations simultaneously: multiply the 1st by 3 and 2nd by 5, then add: $34x=102 \Rightarrow x=3$. Substituting back gives $y=-3$.

Answer: $x = 3$, $y = -3$

Q21. Find the real and imaginary parts of $z = (1+i)/(1-i)$.

[Division + identifying real/imaginary parts — common]

Solution:

Multiply numerator and denominator by the conjugate $(1+i)$:

Numerator: $(1+i)^2 = 1+2i+i^2 = 1+2i-1 = 2i$.

Denominator: $1^2+1^2 = 2$.

$z = 2i/2 = i$. So the real part is 0 and the imaginary part is 1.

Answer: Real part = 0, Imaginary part = 1 ($z = i$)

Q22. If $z = x+iy$, write down the modulus and the conjugate of z in terms of x and y.

[Basic definitions — direct recall question]

Solution:

Modulus: $|z| = \sqrt{x^2+y^2}$.

Conjugate: $\bar{z} = x - iy$.

Answer: $|z| = \sqrt{x^2+y^2}$, $\bar{z} = x-iy$

Q23. Prove that the conjugate of the sum of two complex numbers equals the sum of their conjugates: conjugate(z_{1+z2}) = conjugate(z_1) + conjugate(z_2).

[Standard proof-type question]

Solution:

Let z_1=a+bi and z_2=c+di.

z_{1+z2} = (a+c)+(b+d)i, so conjugate(z_{1+z2}) = (a+c)−(b+d)i.

conjugate(z_1)+conjugate(z_2) = (a−bi)+(c−di) = (a+c)−(b+d)i.

Both sides are equal. (Proved)

Answer: conjugate(z_{1+z2}) = conjugate(z_1)+conjugate(z_2) (proved)

Q24. Solve the quadratic equation $x^2 + 4 = 0$ for its complex roots.

[Quadratic equation with complex roots — very common]

Solution:

$x^2 = -4$.

$x = \pm\sqrt{(-4)} = \pm\sqrt{4}\cdot\sqrt{(-1)} = \pm 2i$.

Answer: x = 2i or x = −2i

Q25. Solve the quadratic equation $x^2 - 2x + 5 = 0$ for its complex roots.

[Quadratic equation with complex roots — very common]

Solution:

Using the quadratic formula: $x = [2 \pm \sqrt{(4-20)}]/2 = [2 \pm \sqrt{(-16)}]/2$.

$= [2 \pm 4i]/2$.

$= 1 \pm 2i$.

Answer: x = 1+2i or x = 1−2i

Q26. Find the value of $(1+i)^2 + (1-i)^2$.

[Direct expansion — common short question]

Solution:

$(1+i)^2 = 1+2i+i^2 = 1+2i-1 = 2i$.

$(1-i)^2 = 1-2i+i^2 = 1-2i-1 = -2i$.

Sum = 2i + (−2i) = 0.

Answer: 0

Q27. If z = 2−3i, find $z + \bar{z}$ and $z\cdot\bar{z}$.

[Property of z and its conjugate — important]

Solution:

$\bar{z} = 2+3i$.

$z + \bar{z} = (2-3i)+(2+3i) = 4$ (equal to twice the real part).

$z \cdot \bar{z} = (2-3i)(2+3i) = 4-9i^2 = 4+9 = 13$ (equal to $|z|^2$).

Answer: $z+\bar{z} = 4$, $z \cdot \bar{z} = 13$

Q28. Find the distance between the points representing $z_1=3+4i$ and $z_2=-1+i$ in the Argand plane.

[Distance in Argand plane — important application]

Solution:

Distance = $|z_1 - z_2|$.

$z_1-z_2 = (3-(-1)) + (4-1)i = 4+3i$.

$|4+3i| = \sqrt{4^2+3^2} = \sqrt{25} = 5$.

Answer: Distance = 5 units

Q29. If $|z-3| = 5$ represents a circle in the Argand plane, find its center and radius.

[Locus in Argand plane — conceptual question]

Solution:

$|z-3|=5$ means the distance of z from the fixed point 3 (i.e. (3,0)) is always 5.

This is the equation of a circle with center (3,0) and radius 5.

Answer: Center = (3,0), Radius = 5

Q30. Simplify: $i^1 + i^2 + i^3 + i^4$.

[Basic — sum of consecutive powers of i]

Solution:

$i^1=i$, $i^2=-1$, $i^3=-i$, $i^4=1$.

Sum = $i + (-1) + (-i) + 1$.

$= (i-i) + (-1+1) = 0$.

Answer: 0
