Chapter: INVERSE CIRCULAR FUNCTION

1. Introduction

An Inverse Circular Function (also called an inverse trigonometric function or arc function) is the inverse of a trigonometric function, restricted to a suitable domain so that the inverse actually exists as a function. If $\sin\theta = x$, then θ is called the inverse sine of x, written $\theta = \sin^{-1}x$ (read as 'arc sine x' or 'angle whose sine is x'). Since trigonometric functions are periodic (many angles give the same ratio), each inverse function is defined only on a restricted range called its PRINCIPAL VALUE BRANCH, so that exactly one output angle corresponds to each input value. This topic is essential for solving trigonometric equations and appears frequently as both direct-evaluation and identity-proof questions.

2. Definition and Notation

- **If $\sin\theta = x$, then $\theta = \sin^{-1}x$ (also written arcsin x)**

Similarly $\cos^{-1}x$, $\tan^{-1}x$, $\cot^{-1}x$, $\sec^{-1}x$, $\text{cosec}^{-1}x$ are defined as the inverses of $\cos\theta$, $\tan\theta$, $\cot\theta$, $\sec\theta$, $\text{cosec}\theta$ respectively.

3. Domain and Principal Value Range

- $\sin^{-1}x$: Domain $[-1,1]$, Range $[-\pi/2, \pi/2]$
- $\cos^{-1}x$: Domain $[-1,1]$, Range $[0, \pi]$
- $\tan^{-1}x$: Domain $\mathbb{R}$(all reals), Range $(-\pi/2, \pi/2)$
- $\cot^{-1}x$: Domain $\mathbb{R}$(all reals), Range $(0, \pi)$
- $\sec^{-1}x$: Domain $|x|\geq 1$, Range $[0,\pi] - \{\pi/2\}$
- $\text{cosec}^{-1}x$: Domain $|x|\geq 1$, Range $[-\pi/2,\pi/2] - \{0\}$

4. Complementary and Odd/Even Relations

- **$\sin^{-1}x + \cos^{-1}x = \pi/2$**
- **$\tan^{-1}x + \cot^{-1}x = \pi/2$**
- **$\sec^{-1}x + \text{cosec}^{-1}x = \pi/2$**
- **$\sin^{-1}(-x) = -\sin^{-1}x$, $\tan^{-1}(-x) = -\tan^{-1}x$ (odd functions)**
- **$\cos^{-1}(-x) = \pi - \cos^{-1}x$ (NOT odd — this special rule applies)**

5. Addition and Multiple-Angle Formulas

- **$\tan^{-1}x + \tan^{-1}y = \tan^{-1}[(x+y)/(1-xy)]$ (when $xy<1$)**
- **$\tan^{-1}x - \tan^{-1}y = \tan^{-1}[(x-y)/(1+xy)]$**

- **$2\tan^{-1}x = \tan^{-1}[2x/(1-x^2)]$ $(|x|<1)$**
- **$\sin^{-1}x + \sin^{-1}y = \sin^{-1}[x\sqrt{1-y^2} + y\sqrt{1-x^2}]$**
- **$\cos^{-1}x + \cos^{-1}y = \cos^{-1}[xy - \sqrt{1-x^2}\sqrt{1-y^2}]$**

6. Important Points for Exams

- Always give the answer for a principal value in RADIANS (unless degrees are explicitly asked), and make sure it lies within the correct range for that specific inverse function.
- When applying $\tan^{-1}x+\tan^{-1}y=\tan^{-1}[(x+y)/(1-xy)]$, if $xy>1$ and both x,y are positive, add π to the result (as shown in the $\tan^{-1}1+\tan^{-1}2+\tan^{-1}3=\pi$ example) — this is a common trap.
- For composite expressions like $\sin[\cos^{-1}x]$, always convert using a right triangle: draw the triangle for the inner inverse function first, then read off the required outer ratio.
- Always check that a value found while solving an inverse-trig equation actually lies within the required principal value range — extraneous roots are common in these problems.
- Remember $\cos^{-1}(-x) = \pi-\cos^{-1}x$ is different from the other (odd-function) rules — this is one of the most common mistakes students make.

7. 30 Important Questions & Answers (With Step-by-Step Solutions)

The following questions are compiled based on the WBSCTE Mathematics I syllabus for the Inverse Circular Function chapter and repeatedly tested question patterns seen in external examination papers. Practising all of these thoroughly gives very strong exam coverage.

Q1. Find the principal value of $\sin^{-1}(1/2)$.

[Basic principal value — asked almost every year]

Solution:

We need θ such that $\sin\theta = 1/2$, with θ in the principal range $[-\pi/2, \pi/2]$.

$\theta = \pi/6$, since $\sin(\pi/6) = 1/2$.

Answer: $\sin^{-1}(1/2) = \pi/6$

Q2. Find the principal value of $\cos^{-1}(\sqrt{3}/2)$.

[Basic principal value — asked almost every year]

Solution:

We need θ such that $\cos\theta = \sqrt{3}/2$, with θ in the principal range $[0, \pi]$.

$\theta = \pi/6$, since $\cos(\pi/6) = \sqrt{3}/2$.

Answer: $\cos^{-1}(\sqrt{3}/2) = \pi/6$

Q3. Find the principal value of $\tan^{-1}(1)$.

[Basic principal value — asked almost every year]

Solution:

We need θ such that $\tan\theta = 1$, with θ in the principal range $(-\pi/2, \pi/2)$.

$\theta = \pi/4$, since $\tan(\pi/4) = 1$.

Answer: $\tan^{-1}(1) = \pi/4$

Q4. Find the principal value of $\sin^{-1}(-1/2)$.

[Principal value with negative input — important]

Solution:

We need θ in $[-\pi/2, \pi/2]$ such that $\sin\theta = -1/2$.

$\theta = -\pi/6$, since $\sin(-\pi/6) = -1/2$.

Answer: $\sin^{-1}(-1/2) = -\pi/6$

Q5. Find the principal value of $\cos^{-1}(-1/2)$.

[Principal value with negative input — important]

Solution:

We need θ in $[0, \pi]$ such that $\cos\theta = -1/2$.

$\theta = 2\pi/3$, since $\cos(2\pi/3) = -1/2$.

Answer: $\cos^{-1}(-1/2) = 2\pi/3$

Q6. Find the principal value of $\tan^{-1}(-1)$.

[Principal value with negative input — important]

Solution:

We need θ in $(-\pi/2, \pi/2)$ such that $\tan\theta = -1$.

$\theta = -\pi/4$, since $\tan(-\pi/4) = -1$.

Answer: $\tan^{-1}(-1) = -\pi/4$

Q7. Find the principal value of $\operatorname{cosec}^{-1}(2)$.

[Principal value of a reciprocal inverse function — common]

Solution:

$\operatorname{cosec}\theta = 2$ means $\sin\theta = 1/2$, with θ in $[-\pi/2, \pi/2]-\{0\}$.

$\theta = \pi/6$.

Answer: $\operatorname{cosec}^{-1}(2) = \pi/6$

Q8. Find the principal value of $\sec^{-1}(2)$.

[Principal value of a reciprocal inverse function — common]

Solution:

$\sec\theta = 2$ means $\cos\theta = 1/2$, with θ in $[0, \pi]-\{\pi/2\}$.

$\theta = \pi/3$.

Answer: $\sec^{-1}(2) = \pi/3$

Q9. Find the principal value of $\cot^{-1}(1)$.

[Principal value of a reciprocal inverse function — common]

Solution:

$\cot\theta = 1$ means $\tan\theta = 1$, with θ in $(0, \pi)$.

$\theta = \pi/4$.

Answer: $\cot^{-1}(1) = \pi/4$

Q10. Find the principal value of $\cot^{-1}(-1)$.

[Principal value with negative input — important]

Solution:

We need θ in $(0, \pi)$ such that $\cot\theta = -1$.

$\theta = 3\pi/4$, since $\cot(3\pi/4) = -1$.

Answer: $\cot^{-1}(-1) = 3\pi/4$

Q11. Verify that $\sin^{-1}x + \cos^{-1}x = \pi/2$, for $x = 1/2$.

[Standard identity verification — very important]

Solution:

$\sin^{-1}(1/2) = \pi/6$. $\cos^{-1}(1/2) = \pi/3$.

Sum $= \pi/6 + \pi/3 = \pi/6 + 2\pi/6 = 3\pi/6 = \pi/2$. (Verified)

Answer: Sum = $\pi/2$ (verified)

Q12. Verify that $\tan^{-1}x + \cot^{-1}x = \pi/2$, for $x = \sqrt{3}$.

[Standard identity verification — very important]

Solution:

$\tan^{-1}(\sqrt{3}) = \pi/3$. $\cot^{-1}(\sqrt{3}) = \pi/6$ (since $\cot(\pi/6) = \sqrt{3}$).

Sum $= \pi/3 + \pi/6 = 2\pi/6 + \pi/6 = 3\pi/6 = \pi/2$. (Verified)

Answer: Sum = $\pi/2$ (verified)

Q13. Using the formula $\tan^{-1}x + \tan^{-1}y = \tan^{-1}[(x+y)/(1-xy)]$, evaluate $\tan^{-1}(1/2) + \tan^{-1}(1/3)$.

[Addition formula application — very important]

Solution:

$= \tan^{-1}[\ (1/2+1/3) / (1-(1/2)(1/3))\ ] = \tan^{-1}[\ (5/6) / (1-1/6)\ ]$.

$= \tan^{-1}[\ (5/6) / (5/6)\ ] = \tan^{-1}(1)$.

$= \pi/4$.

Answer: $\pi/4$

Q14. Prove that $\tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3 = \pi$.

[Classic important identity — asked frequently]

Solution:

Since $2\times3=6>1$, first combine $\tan^{-1}2+\tan^{-1}3$ with the correction term $+\pi$ (as the sum of two obtuse-leaning positive angles exceeds $\pi/2$):

$\tan^{-1}2+\tan^{-1}3 = \pi + \tan^{-1}[(2+3)/(1-6)] = \pi + \tan^{-1}(5/-5) = \pi + \tan^{-1}(-1) = \pi - \pi/4 = 3\pi/4$.

Now add $\tan^{-1}1$: $\tan^{-1}1 + 3\pi/4 = \pi/4 + 3\pi/4 = \pi$. (Proved)

Answer: $\tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3 = \pi$ **(proved)**

Q15. Prove that $2\tan^{-1}(1/3) = \tan^{-1}(3/4)$.

[Multiple-angle inverse-tangent formula application — important]

Solution:

Using $2\tan^{-1}x = \tan^{-1}[2x/(1-x^2)]$ with $x=1/3$:

$= \tan^{-1}[\,(2/3) / (1-1/9)\,] = \tan^{-1}[\,(2/3) / (8/9)\,] = \tan^{-1}[\,(2/3)\times(9/8)\,]$.

$= \tan^{-1}(18/24) = \tan^{-1}(3/4)$. (Proved)

Answer: $2\tan^{-1}(1/3) = \tan^{-1}(3/4)$ (proved)

Q16. Using the formula $\sin^{-1}x + \sin^{-1}y = \sin^{-1}[x\sqrt{(1-y^2)} + y\sqrt{(1-x^2)}]$, evaluate $\sin^{-1}(3/5) + \sin^{-1}(8/17)$.

[Addition formula for $\sin^{-1}$ — very important]

Solution:

$x=3/5$, $y=8/17$. $\sqrt{(1-y^2)} = \sqrt{(1-64/289)} = \sqrt{(225/289)} = 15/17$. $\sqrt{(1-x^2)} = \sqrt{(1-9/25)} = 4/5$.

$x\sqrt{(1-y^2)}+y\sqrt{(1-x^2)} = (3/5)(15/17) + (8/17)(4/5) = 45/85 + 32/85 = 77/85$.

$\sin^{-1}(3/5)+\sin^{-1}(8/17) = \sin^{-1}(77/85)$.

Answer: $\sin^{-1}(77/85)$

Q17. Using the formula $\cos^{-1}x + \cos^{-1}y = \cos^{-1}[xy - \sqrt{(1-x^2)}\sqrt{(1-y^2)}]$, evaluate $\cos^{-1}(4/5) + \cos^{-1}(12/13)$.

[Addition formula for $\cos^{-1}$ — very important]

Solution:

$x=4/5$, $y=12/13$. $\sqrt{(1-x^2)}=3/5$, $\sqrt{(1-y^2)}=5/13$.

$xy - \sqrt{(1-x^2)}\sqrt{(1-y^2)} = (48/65) - (3/5)(5/13) = 48/65 - 15/65 = 33/65$.

$\cos^{-1}(4/5)+\cos^{-1}(12/13) = \cos^{-1}(33/65)$.

Answer: $\cos^{-1}(33/65)$

Q18. Solve for x: $\sin^{-1}x = \pi/6$.

[Basic equation — direct inversion]

Solution:

Taking sine on both sides: $x = \sin(\pi/6)$.

$x = 1/2$.

Answer: $x = 1/2$

Q19. Solve for x: $2\sin^{-1}x = \pi/3$.

[Basic equation — common]

Solution:

$\sin^{-1}x = \pi/6$.

$x = \sin(\pi/6) = 1/2$.

Answer: $x = 1/2$

Q20. Solve for x: $\tan^{-1}(2x) + \tan^{-1}(3x) = \pi/4$.

[Classic important equation using the addition formula]

Solution:

Using the addition formula: $\tan^{-1}[(2x+3x)/(1-6x^2)] = \pi/4$.

$(5x)/(1-6x^2) = \tan(\pi/4) = 1 \implies 5x = 1-6x^2 \implies 6x^2+5x-1 = 0$.

Factorising: $(6x-1)(x+1) = 0 \implies x = 1/6$ or $x = -1$.

Checking validity: $x=-1$ makes both terms negative angles, whose sum cannot equal the positive angle $\pi/4$ — rejected. So $x=1/6$ is the valid solution.

Answer: $x = 1/6$

Q21. Express $\sin^{-1}x$ in terms of $\cos^{-1}x$.

[Basic identity recall]

Solution:

From the complementary relation $\sin^{-1}x + \cos^{-1}x = \pi/2$:

$\sin^{-1}x = \pi/2 - \cos^{-1}x$.

Answer: $\sin^{-1}x = \pi/2 - \cos^{-1}x$

Q22. Express $\tan^{-1}x$ in terms of $\cot^{-1}x$.

[Basic identity recall]

Solution:

From the complementary relation $\tan^{-1}x + \cot^{-1}x = \pi/2$:

$\tan^{-1}x = \pi/2 - \cot^{-1}x$.

Answer: $\tan^{-1}x = \pi/2 - \cot^{-1}x$

Q23. State the domain and (principal value) range of $\sin^{-1}x$.

[Basic definition — must memorise]

Solution:

Domain: $-1 \le x \le 1$, i.e. $[-1,1]$.

Range: $[-\pi/2, \pi/2]$.

Answer: Domain = $[-1,1]$, Range = $[-\pi/2, \pi/2]$

Q24. State the domain and (principal value) range of $\cos^{-1}x$.

[Basic definition — must memorise]

Solution:

Domain: $-1 \le x \le 1$, i.e. $[-1,1]$.

Range: $[0, \pi]$.

Answer: Domain = $[-1,1]$, Range = $[0, \pi]$

Q25. State the domain and (principal value) range of $\tan^{-1}x$.

[Basic definition — must memorise]

Solution:

Domain: all real numbers, i.e. $(-\infty, \infty)$.

Range: $(-\pi/2, \pi/2)$.

Answer: Domain = $\mathbb{R}$ Range = $(-\pi/2, \pi/2)$

Q26. Verify that $\sin^{-1}x = \text{cosec}^{-1}(1/x)$, for $x = 1/2$.

[Reciprocal-relation verification — important]

Solution:

$\sin^{-1}(1/2) = \pi/6$.

$\text{cosec}^{-1}(1/(1/2)) = \text{cosec}^{-1}(2) = \pi/6$ (since $\text{cosec}(\pi/6)=2$).

Both sides equal $\pi/6$. (Verified)

Answer: Both equal $\pi/6$ (verified)

Q27. Evaluate: $\sin[\cos^{-1}(3/5)]$.

[Composite (mixed) function evaluation — very important]

Solution:

Let $\theta = \cos^{-1}(3/5)$, so $\cos\theta = 3/5$, with θ in $[0,\pi]$ and $\sin\theta \geq 0$.

$\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-9/25} = \sqrt{16/25} = 4/5$.

Answer: $\sin[\cos^{-1}(3/5)] = 4/5$

Q28. Evaluate: $\cos[\tan^{-1}(3/4)]$.

[Composite (mixed) function evaluation — very important]

Solution:

Let $\theta = \tan^{-1}(3/4)$, so $\tan\theta = 3/4$, with θ acute (positive input, principal range).

Using a right triangle: opposite=3, adjacent=4, hypotenuse=$\sqrt{3^2+4^2}$=5.

$\cos\theta$ = adjacent/hypotenuse = 4/5.

Answer: $\cos[\tan^{-1}(3/4)] = 4/5$

Q29. Evaluate: $\tan[\sin^{-1}(5/13)]$.

[Composite (mixed) function evaluation — very important]

Solution:

Let $\theta = \sin^{-1}(5/13)$, so $\sin\theta = 5/13$, θ acute.

$\cos\theta = \sqrt{(1-\sin^2\theta)} = \sqrt{(1-25/169)} = \sqrt{(144/169)} = 12/13$.

$\tan\theta = \sin\theta/\cos\theta = (5/13)/(12/13) = 5/12$.

Answer: $\tan[\sin^{-1}(5/13)] = 5/12$

Q30. Prove that $\tan^{-1}(1/2) + \tan^{-1}(1/5) + \tan^{-1}(1/8) = \pi/4$.

[Classic important closing identity — uses addition formula twice]

Solution:

Step 1: $\tan^{-1}(1/2)+\tan^{-1}(1/5) = \tan^{-1}[(1/2+1/5)/(1-(1/2)(1/5))] = \tan^{-1}[(7/10)/(9/10)] = \tan^{-1}(7/9)$.

Step 2: $\tan^{-1}(7/9)+\tan^{-1}(1/8) = \tan^{-1}[(7/9+1/8)/(1-(7/9)(1/8))] = \tan^{-1}[(65/72)/(65/72)] = \tan^{-1}(1)$.

$= \pi/4$. (Proved)

Answer: $\tan^{-1}(1/2)+\tan^{-1}(1/5)+\tan^{-1}(1/8) = \pi/4$ (proved)
