Chapter: LIMIT

1. Introduction

The concept of a Limit is the foundation of all Differential Calculus — it describes the value that a function f(x) approaches as the input x gets closer and closer to some value 'a', without necessarily ever reaching it. This chapter of the WBSCTE Mathematics syllabus covers the definition of a limit, left-hand and right-hand limits, algebra of limits, a set of standard limits that must be memorised, and the closely related idea of continuity of a function at a point. Almost every question in this chapter is a direct application of a standard limit or a simple algebraic technique (factorization or rationalization), making it very scoring if the formulas are well practised.

2. Definition of a Limit

We say that l is the limit of f(x) as x approaches a, written:

- **$\lim_{x\to a} f(x) = l$**

if the value of f(x) can be made as close to l as we like by taking x sufficiently close to a (but not equal to a).

3. Left-Hand Limit and Right-Hand Limit

- Left-Hand Limit (LHL): the value f(x) approaches as x approaches a from values LESS than a, written $\lim_{x\to a^-} f(x)$.
- Right-Hand Limit (RHL): the value f(x) approaches as x approaches a from values GREATER than a, written $\lim_{x\to a^+} f(x)$.
- The limit $\lim_{x\to a} f(x)$ EXISTS if and only if LHL = RHL.

4. Algebra of Limits

If $\lim_{x\to a} f(x) = l$ and $\lim_{x\to a} g(x) = m$, then:

- **$\lim_{x\to a} [f(x) \pm g(x)] = l \pm m$**
- **$\lim_{x\to a} [f(x) \cdot g(x)] = l \cdot m$**
- **$\lim_{x\to a} [f(x) / g(x)] = l/m$ (provided $m \neq 0$)**

5. Standard Limits (Very Important — Memorise All)

- **$\lim_{x\to a} (x^n - a^n)/(x - a) = n \cdot a^{n-1}$**
- **$\lim_{x\to 0} \sin x / x = 1$**
- **$\lim_{x\to 0} \tan x / x = 1$**

- **$\lim_{x\to0} (1-\cos x)/x^2 = 1/2$**
- **$\lim_{x\to0} (e^x-1)/x = 1$**
- **$\lim_{x\to0} (a^x-1)/x = \ln a$**
- **$\lim_{x\to0} \log(1+x)/x = 1$**
- **$\lim_{x\to0} (1+x)^{1/x} = e$**
- **$\lim_{x\to\infty} (1+1/x)^x = e$, $\lim_{x\to\infty} (1+k/x)^x = e^k$**

6. Continuity of a Function at a Point

A function f(x) is said to be continuous at x = a if all three of the following conditions hold:

- f(a) is defined,
- $\lim_{x\to a} f(x)$ exists (i.e. LHL = RHL),
- $\lim_{x\to a} f(x) = f(a)$.

If any one of these fails, f(x) is said to be discontinuous at x = a.

7. Important Points for Exams

- If direct substitution gives the indeterminate form 0/0, always try factorization (for polynomials) or rationalization (for square roots) before anything else.
- For limits at infinity of rational functions, divide the numerator and denominator by the highest power of x present in the denominator.
- Standard limits (sinx/x, $(e^x-1)/x$, etc.) are ONLY valid in the form shown — always adjust the coefficient of x in the argument to match the denominator exactly, as shown in the multiple-angle examples.
- For continuity questions with an unknown constant k, always set $\lim_{x\to a} f(x) = f(a)$ and solve for k.
- A limit failing to exist (LHL ≠ RHL) is different from a function being undefined at a point — both are tested separately and are common exam traps.

8. 30 Important Questions & Answers (With Step-by-Step Solutions)

The following questions are compiled based on the WBSCTE Mathematics syllabus for the Limit chapter and repeatedly tested question patterns seen in external examination papers. Practising all of these thoroughly gives very strong exam coverage.

Q1. Evaluate: $\lim_{x\to2} (x^2+3x-1)$.

[Basic — direct substitution]

Solution:

Since x^2+3x-1 is a polynomial, it is continuous everywhere, so we can substitute $x=2$ directly.

$= (2)^2+3(2)-1 = 4+6-1$.

$= 9$.

Answer: 9

Q2. Evaluate: $\lim_{x\to3} (x^2-9)/(x-3)$.

[0/0 form — factorization method — very important]

Solution:

Direct substitution gives 0/0, an indeterminate form, so we factorize first.

$(x^2-9)/(x-3) = (x-3)(x+3)/(x-3) = (x+3)$, for $x\neq3$.

$\lim_{x\to3} (x+3) = 3+3 = 6$.

Answer: 6

Q3. Evaluate: $\lim_{x\to1} (x^3-1)/(x-1)$.

[0/0 form — factorization method — very important]

Solution:

$x^3-1 = (x-1)(x^2+x+1)$ [using $a^3-b^3 = (a-b)(a^2+ab+b^2)$].

$(x^3-1)/(x-1) = x^2+x+1$, for $x\neq1$.

$\lim_{x\to1} (x^2+x+1) = 1+1+1 = 3$.

Answer: 3

Q4. Evaluate: $\lim_{x\to0} (\sqrt{(1+x)} - 1)/x$.

[0/0 form — rationalization method — very important]

Solution:

Multiply numerator and denominator by the conjugate $(\sqrt{(1+x)}+1)$:

$= [(1+x)-1] / [x(\sqrt{(1+x)}+1)] = x / [x(\sqrt{(1+x)}+1)] = 1/(\sqrt{(1+x)}+1)$.

$\lim_{x\to0} 1/(\sqrt{(1+x)}+1) = 1/(1+1) = 1/2$.

Answer: 1/2

Q5. Evaluate: $\lim_{x\to4}$ $(\sqrt{x} - 2)/(x-4)$.

[0/0 form — rationalization method — very important]

Solution:

Multiply numerator and denominator by $(\sqrt{x}+2)$:

$= (x-4) / [(x-4)(\sqrt{x}+2)] = 1/(\sqrt{x}+2)$, for $x\neq4$.

$\lim_{x\to4} 1/(\sqrt{x}+2) = 1/(2+2) = 1/4$.

Answer: 1/4

Q6. Evaluate: $\lim_{x\to0}$ $\sin5x / x$.

[Standard limit $\sin\vartheta/\vartheta = 1$ — asked almost every year]

Solution:

Write $\sin5x/x = [\sin5x/5x] \times 5$.

Using the standard limit $\lim_{\theta\to0} \sin\theta/\theta = 1$, with $\theta=5x$: $\lim_{x\to0} \sin5x/5x = 1$.

So the required limit $= 1 \times 5 = 5$.

Answer: 5

Q7. Evaluate: $\lim_{x\to0}$ $\sin3x / \sin5x$.

[Standard limit ratio type — very important]

Solution:

$\sin3x/\sin5x = [\sin3x/3x] \times [5x/\sin5x] \times (3x/5x)$.

As $x\to0$: $\sin3x/3x \to 1$ and $5x/\sin5x \to 1$, leaving $3x/5x = 3/5$.

So the limit $= 1 \times 1 \times 3/5 = 3/5$.

Answer: 3/5

Q8. Evaluate: $\lim_{x\to0}$ $\tan4x / x$.

[Standard limit $\tan\vartheta/\vartheta = 1$ — very important]

Solution:

Write $\tan4x/x = [\tan4x/4x] \times 4$.

Using $\lim_{\theta\to0} \tan\theta/\theta = 1$ with $\theta=4x$: $\lim_{x\to0} \tan4x/4x = 1$.

So the limit $= 1 \times 4 = 4$.

Answer: 4

Q9. Evaluate: $\lim_{x\to0}$ $(1-\cos x)/x^2$.

[Standard limit $(1-\cos\vartheta)/\vartheta^2 = 1/2$ — very important]

Solution:

This is a standard limit result: $\lim_{x\to0} (1-\cos x)/x^2 = 1/2$.

(Derivation uses $1-\cos x = 2\sin^2(x/2)$, giving $2\sin^2(x/2)/x^2 = (1/2)[\sin(x/2)/(x/2)]^2$, which → 1/2 as $x \to 0$.)

Answer: 1/2

Q10. Evaluate: $\lim_{x\to 0} (1-\cos 2x)/x^2$.

[Standard limit applied with a multiple angle — important]

Solution:

$1-\cos 2x = 2\sin^2 x$, so $(1-\cos 2x)/x^2 = 2\sin^2 x/x^2 = 2[\sin x/x]^2$.

$\lim_{x\to 0} [\sin x/x]^2 = 1^2 = 1$.

So the limit = 2 × 1 = 2.

Answer: 2

Q11. Using the standard limit $\lim_{x\to a} (x^n-a^n)/(x-a) = n \cdot a^{n-1}$, evaluate $\lim_{x\to 2} (x^5-32)/(x-2)$.

[Standard limit — very important formula-based question]

Solution:

Here n=5, a=2, so $a^n = 2^5 = 32$, matching the numerator form.

Applying the formula: $n \cdot a^{n-1} = 5 \cdot 2^4 = 5 \times 16$.

= 80.

Answer: 80

Q12. Evaluate: $\lim_{x\to 0} (e^x - 1)/x$.

[Standard exponential limit — very important]

Solution:

This is a standard result: $\lim_{x\to 0} (e^x-1)/x = 1$.

Answer: 1

Q13. Evaluate: $\lim_{x\to 0} (e^{3x} - 1)/x$.

[Standard exponential limit with multiple of x — important]

Solution:

Write $(e^{3x}-1)/x = [(e^{3x}-1)/3x] \times 3$.

Using $\lim_{x\to 0} (e^u-1)/u = 1$ with u=3x: $\lim_{x\to 0} (e^{3x}-1)/3x = 1$.

So the limit = 1 × 3 = 3.

Answer: 3

Q14. State the standard limit $\lim_{x\to 0} (a^x-1)/x$, and hence evaluate $\lim_{x\to 0} (5^x-1)/x$.

[Standard exponential limit (general base) — important]

Solution:

Standard result: $\lim_{x\to0} (a^x-1)/x = \ln a$ (natural log of a), for $a > 0$.

Applying with a=5: $\lim_{x\to0} (5^x-1)/x = \ln 5$.

Answer: ln 5

Q15. Evaluate: $\lim_{x\to0} \log(1+x) / x$.

[Standard logarithmic limit — very important]

Solution:

This is a standard result: $\lim_{x\to0} \log(1+x)/x = 1$.

Answer: 1

Q16. Evaluate: $\lim_{x\to0} (1+x)^{1/x}$.

[Standard limit — definition of e — very important]

Solution:

This is the standard limit that defines the number e:

$\lim_{x\to0} (1+x)^{1/x} = e$.

Answer: e

Q17. Evaluate: $\lim_{x\to\infty} (1 + 3/x)^x$.

[Standard limit — exponential form e^k — important]

Solution:

This uses the standard result: $\lim_{x\to\infty} (1+k/x)^x = e^k$.

Here k=3, so the limit = e^3.

Answer: e^3

Q18. Evaluate: $\lim_{x\to\infty} (2x^2+3x+1) / (4x^2-x+5)$.

[Limit at infinity — same degree — very important]

Solution:

Divide numerator and denominator by the highest power of x, which is x^2:

$= (2 + 3/x + 1/x^2) / (4 - 1/x + 5/x^2)$.

As $x\to\infty$, all terms with $1/x$ and $1/x^2 \to 0$, leaving 2/4.

= 1/2.

Answer: 1/2

Q19. Evaluate: $\lim_{x\to\infty} (3x+5) / (x^2+1)$.

[Limit at infinity — denominator degree higher — common]

Solution:

Divide numerator and denominator by x^2: $= (3/x + 5/x^2) / (1 + 1/x^2)$.

As $x \to \infty$, the numerator $\to 0$ and the denominator $\to 1$.

So the limit = 0.

Answer: 0

Q20. Evaluate: $\lim_{x\to\infty} (x^2+1) / (3x+5)$.

[Limit at infinity — numerator degree higher — common]

Solution:

Divide numerator and denominator by x: $= (x + 1/x) / (3 + 5/x)$.

As $x \to \infty$, the numerator grows without bound while the denominator tends to 3.

So the limit does not exist as a finite number; it tends to infinity.

Answer: Limit → ∞ (does not exist as a finite value)

Q21. Discuss the continuity of $f(x) = x^2 - 5x + 6$ at $x = 2$.

[Basic continuity check — polynomial function]

Solution:

A function f is continuous at x=a if $\lim_{x\to a} f(x) = f(a)$.

$f(2) = 4 - 10 + 6 = 0$.

Since f(x) is a polynomial, $\lim_{x\to 2} f(x) = f(2) = 0$ automatically.

Since the limit equals the function value, f is continuous at x=2.

Answer: f(x) is continuous at x=2 (since limit = f(2) = 0)

Q22. Examine whether $f(x) = (x^2-4)/(x-2)$ is continuous at $x = 2$.

[Removable discontinuity — important conceptual question]

Solution:

f(x) is undefined at x=2, since the denominator becomes zero there.

However, $\lim_{x\to 2} (x^2-4)/(x-2) = \lim_{x\to 2} (x+2) = 4$ (the limit exists).

Since f(2) is not defined, the condition $\lim_{x\to 2} f(x) = f(2)$ cannot be checked/satisfied.

Hence f is NOT continuous at x=2 (it has a removable discontinuity there).

Answer: f is discontinuous at x=2 (removable discontinuity, though the limit=4 exists)

Q23. If $f(x) = (x^2-1)/(x-1)$ for $x\neq 1$, and $f(x)=k$ for $x=1$, find k so that f is continuous at x=1.

[Classic important question — finding unknown for continuity]

Solution:

For continuity at x=1: $\lim_{x\to 1} f(x) = f(1) = k$.

$\lim_{x\to 1} (x^2-1)/(x-1) = \lim_{x\to 1} (x-1)(x+1)/(x-1) = \lim_{x\to 1} (x+1) = 2$.

So k must equal 2.

Answer: k = 2

Q24. Find the left-hand and right-hand limits of $f(x) = |x|/x$ at $x = 0$, and state whether $\lim_{x\to0} f(x)$ exists.

[Left-hand/right-hand limit — very important conceptual question]

Solution:

For $x > 0$: $|x| = x$, so $f(x) = x/x = 1$. Hence the Right-Hand Limit (RHL) = $\lim_{x\to0^+} f(x) = 1$.

For $x < 0$: $|x| = -x$, so $f(x) = -x/x = -1$. Hence the Left-Hand Limit (LHL) = $\lim_{x\to0^-} f(x) = -1$.

Since LHL (-1) ≠ RHL (1), the limit $\lim_{x\to0} f(x)$ does NOT exist.

Answer: LHL = −1, RHL = 1; the limit does not exist at x=0

Q25. Using the definition of a limit, evaluate $\lim_{h\to0} [(x+h)^2 - x^2] / h$.

[First-principles style limit — connects to derivative concept — important]

Solution:

$(x+h)^2 - x^2 = x^2+2xh+h^2 - x^2 = 2xh + h^2$.

$[(x+h)^2-x^2]/h = (2xh+h^2)/h = 2x+h$, for $h\neq0$.

$\lim_{h\to0} (2x+h) = 2x$.

Answer: 2x

Q26. Evaluate: $\lim_{x\to0} (e^x - e^{-x}) / x$.

[Combined exponential-limit application — important]

Solution:

Write $(e^x-e^{-x})/x = (e^x-1)/x - (e^{-x}-1)/x$.

$\lim_{x\to0} (e^x-1)/x = 1$.

For the second term, let $u=-x$; as $x\to0$, $(e^{-x}-1)/x = -(e^u-1)/u \to -1$, so $(e^{-x}-1)/x \to -1$.

Total = $1 - (-1) = 2$.

Answer: 2

Q27. Evaluate: $\lim_{x\to0} (1-\cos x) / (x \sin x)$.

[Combined standard-limits application — important]

Solution:

Write $(1-\cos x)/(x \sin x) = [(1-\cos x)/x^2] \times [x/\sin x]$.

$\lim_{x\to0} (1-\cos x)/x^2 = 1/2$, and $\lim_{x\to0} x/\sin x = 1$.

So the limit = $(1/2) \times 1 = 1/2$.

Answer: 1/2

Q28. Evaluate: $\lim_{x\to1} (x^3-1) / (x^2-1)$.

[0/0 form — factorization, combined cubic/quadratic — common]

Solution:

$x^3-1 = (x-1)(x^2+x+1)$, and $x^2-1 = (x-1)(x+1)$.

$(x^3-1)/(x^2-1) = (x^2+x+1)/(x+1)$, for $x\neq1$.

$\lim_{x\to1} (x^2+x+1)/(x+1) = (1+1+1)/(1+1) = 3/2$.

Answer: 3/2

Q29. Evaluate: $\lim_{x\to0} (\sqrt{4+x} - 2) / x$.

[0/0 form — rationalization method — common]

Solution:

Multiply numerator and denominator by the conjugate $(\sqrt{4+x}+2)$:

$= (4+x-4) / [x(\sqrt{4+x}+2)] = x / [x(\sqrt{4+x}+2)] = 1/(\sqrt{4+x}+2)$, for $x\neq0$.

$\lim_{x\to0} 1/(\sqrt{4+x}+2) = 1/(2+2) = 1/4$.

Answer: 1/4

Q30. If $\lim_{x\to2} (x^2+ax+b)/(x-2)$ exists and is finite, and $a=-5$, find b and the value of the limit.

[Conceptual application — classic important closing question]

Solution:

For the limit of $(x^2+ax+b)/(x-2)$ to exist finitely as $x\to2$ (where the denominator $\to 0$), the numerator must ALSO be zero at $x=2$.

So $4 + 2a + b = 0$. With $a=-5$: $4 - 10 + b = 0 \implies b = 6$.

Numerator becomes $x^2-5x+6 = (x-2)(x-3)$.

$\lim_{x\to2} (x-2)(x-3)/(x-2) = \lim_{x\to2} (x-3) = -1$.

Answer: b = 6, and the value of the limit = −1
