Chapter: LOGARITHM

1. Introduction

A logarithm is the inverse operation of exponentiation (raising a number to a power). If a base 'a' ($a > 0$, $a \neq 1$) is raised to a power x to give a number N ($N > 0$), then x is called the logarithm of N to the base a. In symbols, if $a^x = N$, then $x = \log_a N$. Logarithms convert multiplication into addition, division into subtraction, and powers into products, which makes complicated calculations much simpler — this is why the chapter is important both for exams and for real engineering computation (e.g., decibel scale, pH scale, Richter scale).

2. Definition

If $a^x = N$, where $a > 0$, $a \neq 1$ and $N > 0$, then x is called the logarithm of N to the base a, written as:

- **$x = \log_a N \iff a^x = N$**

Two special systems are commonly used:

- Common logarithm — base 10, written as log N (or $\log_{10}N$).
- Natural logarithm — base e ($e \approx 2.71828$), written as ln N (or $\log_e N$).

3. Fundamental Formulas & Laws (Very Important — Memorise All)

- **$\log_a 1 = 0$ (for any valid base a)**
- **$\log_a a = 1$**
- **Product Law: $\log_a(mn) = \log_a m + \log_a n$**
- **Quotient Law: $\log_a(m/n) = \log_a m - \log_a n$**
- **Power Law: $\log_a(m^n) = n \cdot \log_a m$**
- **Change of Base: $\log_a N = \log_b N / \log_b a = 1 / \log_N a$**
- **$\log_{a^m} a^n = n/m$**
- **If $x = \log_a N$, then $a^x = N$ (fundamental log identity)**
- **$\log_a b \cdot \log_b a = 1$**
- **$\log_a b \cdot \log_b c \cdot \log_c a = 1$**
- **If $\log_a m = \log_a n$, then $m = n$ ($a > 0$, $a \neq 1$)**

4. Important Points for Exams

- Logarithm is defined only for positive numbers with a positive base not equal to 1. Always check the domain after solving log equations.

- $\log(m + n) \neq \log m + \log n$. This is a very common mistake — the product law applies to $\log(m \cdot n)$, NOT $\log(m+n)$.
- Whenever a problem gives a symmetric condition like $\log x/(y-z) = \log y/(z-x) = \log z/(x-y)$, introduce a constant k and proceed — this is the standard technique for WBSCTE 'prove that' questions.
- For change-of-base problems (e.g., $\log_2 x + \log_4 x + \log_{16} x = k$), convert every term to the same base before solving.
- Characteristic and mantissa concepts (number of digits, using log tables) appear almost every year — practice $\log_{10}2$, $\log_{10}3$ based numerical problems.

5. 30 Important Questions & Answers (With Step-by-Step Solutions)

The following questions are compiled based on WBSCTE Mathematics I previous year external examination papers (last several years) and repeatedly tested question patterns for the Logarithm chapter. Practising all of these thoroughly gives very strong exam coverage.

Q1. Define logarithm of a number. Show that $\log_a a = 1$ and $\log_a 1 = 0$.

[Basic definition — asked almost every year]

Solution:

By definition, if $a^x = N$ ($a > 0$, $a \neq 1$, $N > 0$), then $x = \log_a N$.

To show $\log_a a = 1$: Let $\log_a a = x$, so $a^x = a = a^1$, hence $x = 1$.

To show $\log_a 1 = 0$: Let $\log_a 1 = x$, so $a^x = 1 = a^0$, hence $x = 0$.

Answer: $\log_a a = 1$ and $\log_a 1 = 0$.

Q2. Solve for x: $\log_2 x + \log_4 x = 3$.

[Standard base-conversion type (WBSCTE pattern)]

Solution:

$\log_4 x = \log_2 x / \log_2 4 = \log_2 x / 2$ (change of base).

So the equation becomes $\log_2 x + (\log_2 x)/2 = 3$.

$\Rightarrow (3/2) \log_2 x = 3 \Rightarrow \log_2 x = 2$.

$\Rightarrow x = 2^2 = 4$.

Answer: $x = 4$

Q3. If $u(x) = \log \sin x$ and $v(x) = \log \cos x$, find the value of $e^{2u} + e^{2v}$ at $x = \pi/4$.

[WBSCTE 2007, 2009, 2015]

Solution:

$e^{2u} = e^{2 \log \sin x} = e^{\log \sin^2 x} = \sin^2 x$.

Similarly, $e^{2v} = e^{2 \log \cos x} = \cos^2 x$.

So $e^{2u} + e^{2v} = \sin^2 x + \cos^2 x = 1$ (identity holds for all x).

Answer: $e^{2u} + e^{2v} = 1$

Q4. Find the value of $\log_{\sqrt{7}} 343$.

[WBSCTE 2007, 2009, 2011]

Solution:

$343 = 7^3$ and $\sqrt{7} = 7^{1/2}$.

$\log_{\sqrt{7}} 343 = \log_{7^{(1/2)}} 7^3 = 3 / (1/2) = 6$ [using $\log_{a^m} a^n = n/m$]

Answer: $\log_{\sqrt{7}} 343 = 6$

Q5. If $\log_x 81 = 4$, find the value of x.

[WBSCTE pattern (based on 2011 paper)]

Solution:

$\log_x 81 = 4$ means $x^4 = 81$.

$81 = 3^4$, so $x^4 = 3^4$.

Since x must be positive, x = 3.

Answer: x = 3

Q6. Find the value of log tan1° + log tan2° + log tan3° + ... + log tan89°.

[WBSCTE 2012]

Solution:

Pair the terms: $\tan\theta \cdot \tan(90° - \theta) = \tan\theta \cdot \cot\theta = 1$ for $\theta = 1°, 2°, ..., 44°$ paired with 89°,88°,...,46°.

So log tan1° + log tan89° = log(tan1°·tan89°) = log 1 = 0, and similarly for each pair up to log tan44° + log tan46° = 0.

The middle term log tan45° = log 1 = 0 (since tan45° = 1).

Adding all 89 terms (44 pairs, each = 0, plus the middle term = 0) gives the total sum = 0.

Answer: Sum = 0

Q7. Find the value of $\log_2 \log_2 \log_2 16$ (i.e. log base 2, applied three times).

[WBSCTE 2013]

Solution:

First: $\log_2 16 = \log_2 2^4 = 4$.

Second: $\log_2 4 = \log_2 2^2 = 2$.

Third: $\log_2 2 = 1$.

So $\log_2 \log_2 \log_2 16 = 1$.

Answer: Value = 1

Q8. If $\log_2 \log_2 \log_2 x = 1$, find the value of x.

[WBSCTE 2014 pattern]

Solution:

$\log_2 \log_2 \log_2 x = 1 \implies \log_2 \log_2 x = 2^1 = 2$.

$\implies \log_2 x = 2^2 = 4$.

$\implies x = 2^4 = 16$.

Answer: x = 16

Q9. If log x / (y − z) = log y / (z − x) = log z / (x − y), prove that xyz = 1.

[WBSCTE 2005, 2009, 2012]

Solution:

Let log x/(y−z) = log y/(z−x) = log z/(x−y) = k.

So log x = k(y−z), log y = k(z−x), log z = k(x−y).

Add all three: log x + log y + log z = k[(y−z)+(z−x)+(x−y)] = k(0) = 0.

So log(xyz) = 0 = log 1, hence xyz = 1. (Proved)

Answer: xyz = 1 (proved)

Q10. If log x/(y−z) = log y/(z−x) = log z/(x−y), prove that $x^x \cdot y^y \cdot z^z = 1$.

[Frequently asked — long answer type]

Solution:

Let each ratio = k, so log x = k(y−z), log y = k(z−x), log z = k(x−y).

Multiply the first equation by x: x log x = kx(y−z). Similarly y log y = ky(z−x), z log z = kz(x−y).

Add: x log x + y log y + z log z = k[x(y−z) + y(z−x) + z(x−y)] = k[xy−xz+yz−xy+xz−yz] = k(0) = 0.

So $\log(x^x) + \log(y^y) + \log(z^z) = 0$, i.e. $\log(x^x y^y z^z) = 0$.

Hence $x^x \cdot y^y \cdot z^z = 1$. (Proved)

Answer: $x^x y^y z^z = 1$ (proved)

Q11. The logarithm of 729 to the base 3 is ______.

[WBSCTE 2018]

Solution:

$729 = 3^6$ (since 3×3×3×3×3×3 = 729).

$\log_3 729 = \log_3 3^6 = 6 \log_3 3 = 6 \times 1 = 6$.

Answer: $\log_3 729 = 6$

Q12. If $\log_a x^2 = 2$, find the value of x.

[WBSCTE 2017]

Solution:

$\log_a x^2 = 2 \implies 2 \log_a x = 2 \implies \log_a x = 1$.

$\implies x = a^1 = a$.

(For the numeric MCQ version with base fixed, x = 2.)

Answer: x = a (or x = 2 for the fixed-base version)

Q13. State and prove the product law of logarithm: $\log_a(mn) = \log_a m + \log_a n$.

[Core formula — often asked as proof]

Solution:

Let $\log_a m = x$ and $\log_a n = y$, so $m = a^x$ and $n = a^y$.

Then $mn = a^x \cdot a^y = a^{x+y}$.

Taking log base a on both sides: $\log_a(mn) = x + y = \log_a m + \log_a n$. (Proved)

Answer: $\log_a(mn) = \log_a m + \log_a n$ (proved)

Q14. State and prove the quotient law of logarithm: $\log_a(m/n) = \log_a m - \log_a n$.

[Core formula — often asked as proof]

Solution:

Let $\log_a m = x$ and $\log_a n = y$, so $m = a^x$, $n = a^y$.

Then $m/n = a^x/a^y = a^{x-y}$.

Taking log base a: $\log_a(m/n) = x - y = \log_a m - \log_a n$. (Proved)

Answer: $\log_a(m/n) = \log_a m - \log_a n$ (proved)

Q15. State and prove the power law of logarithm: $\log_a(m^n) = n \log_a m$.

[Core formula — often asked as proof]

Solution:

Let $\log_a m = x$, so $m = a^x$.

Then $m^n = (a^x)^n = a^{nx}$.

Taking log base a: $\log_a(m^n) = nx = n \log_a m$. (Proved)

Answer: $\log_a(m^n) = n \log_a m$ (proved)

Q16. Prove the change-of-base rule: $\log_a N = \log_b N / \log_b a$.

[Core formula — often asked]

Solution:

Let $\log_a N = x$, so $a^x = N$.

Take log base b on both sides: $\log_b(a^x) = \log_b N \implies x \log_b a = \log_b N$.

So $x = \log_b N / \log_b a$, i.e. $\log_a N = \log_b N / \log_b a$. (Proved)

Answer: $\log_a N = \log_b N / \log_b a$ (proved)

Q17. Solve for x: $\log_2(x + 1) + \log_2(x - 1) = 3$.

[Standard logarithmic equation type]

Solution:

Combine using product law: $\log_2[(x+1)(x-1)] = 3 \implies \log_2(x^2 - 1) = 3$.

So $x^2 - 1 = 2^3 = 8 \implies x^2 = 9 \implies x = \pm 3$.

Check domain: $x + 1 > 0$ and $x - 1 > 0 \implies x > 1$. So $x = -3$ is rejected.

Answer: $x = 3$

Q18. Solve for x: $\log(x - 3) + \log(x + 3) = \log(x + 21)$ (base 10).

[Standard logarithmic equation type]

Solution:

LHS = $\log[(x-3)(x+3)] = \log(x^2 - 9)$.

So $\log(x^2 - 9) = \log(x + 21) \implies x^2 - 9 = x + 21 \implies x^2 - x - 30 = 0$.

Factorising: $(x - 6)(x + 5) = 0 \implies x = 6$ or $x = -5$.

Domain requires $x - 3 > 0 \implies x > 3$, so $x = -5$ is rejected.

Answer: $x = 6$

Q19. If $\log_{10}2 = 0.3010$ and $\log_{10}3 = 0.4771$, find $\log_{10}12$.

[Standard value-based computation]

Solution:

$12 = 2^2 \times 3$, so $\log 12 = \log 2^2 + \log 3 = 2 \log 2 + \log 3$.

$= 2(0.3010) + 0.4771 = 0.6020 + 0.4771 = 1.0791$.

Answer: $\log_{10}12 = 1.0791$

Q20. If $\log_{10}2 = 0.3010$, find the number of digits in 2^{40}.

[Application of logarithm — common paper question]

Solution:

Let $N = 2^{40}$. Then $\log_{10}N = 40 \log_{10}2 = 40 \times 0.3010 = 12.04$.

Number of digits = $\lfloor \log_{10}N \rfloor + 1 = 12 + 1 = 13$ (since the characteristic is 12).

Answer: 2^{40} has 13 digits

Q21. Solve: $\log_x 8 = 3/2$, find x.

[Direct application of definition]

Solution:

$\log_x 8 = 3/2$ means $x^{3/2} = 8 = 2^3$.

So $x^{3/2} = 2^3 \implies x = 2^{3 \cdot 2/3} = 2^2 = 4$.

Answer: $x = 4$

Q22. If $\log_a b \cdot \log_b c \cdot \log_c a = 1$, verify the identity for $a, b, c > 0$, $a,b,c \neq 1$.

[Standard identity — long answer]

Solution:

Using change of base with a common base (say base 10): $\log_a b = \log b/\log a$, $\log_b c = \log c/\log b$, $\log_c a = \log a/\log c$.

Multiplying: (log b/log a) × (log c/log b) × (log a/log c) = (log a · log b · log c)/(log a · log b · log c) = 1.

Hence the identity $\log_a b \cdot \log_b c \cdot \log_c a = 1$ is verified.

Answer: Identity verified = 1

Q23. Simplify: $\log(a^2/bc) + \log(b^2/ca) + \log(c^2/ab)$.

[Standard simplification — long/short answer]

Solution:

Sum = $\log[(a^2/bc) \times (b^2/ca) \times (c^2/ab)]$ (using product law in reverse).

= $\log[(a^2b^2c^2)/(a^2b^2c^2)] = \log 1 = 0$.

Answer: The expression = 0

Q24. If $x = \log_a(bc)$, $y = \log_b(ca)$, $z = \log_c(ab)$, prove that $1/(1+x) + 1/(1+y) + 1/(1+z) = 1$.

[Frequently seen long-answer identity]

Solution:

$1 + x = 1 + \log_a(bc) = \log_a a + \log_a(bc) = \log_a(abc)$.

So $1/(1+x) = 1/\log_a(abc) = \log_{abc} a$ (change of base).

Similarly $1/(1+y) = \log_{abc} b$ and $1/(1+z) = \log_{abc} c$.

Adding: $\log_{abc} a + \log_{abc} b + \log_{abc} c = \log_{abc}(abc) = 1$. (Proved)

Answer: Sum = 1 (proved)

Q25. Find x if 2 log x = log 2 + log 18.

[Common short-answer question]

Solution:

RHS = log(2 × 18) = log 36.

LHS = $\log x^2$.

So $\log x^2 = \log 36 \implies x^2 = 36 \implies x = \pm 6$.

Since x must be positive for log x to be defined, x = 6.

Answer: x = 6

Q26. Prove: log(1 + 2 + 3) = log1 + log2 + log3.

[Conceptual / trick question — tests understanding of log laws]

Solution:

LHS: log(1+2+3) = log 6.

RHS: log1 + log2 + log3 = 0 + log2 + log3 = log(2×3) = log 6.

Since LHS = RHS = log6, the equality holds here — but note this is a coincidence for these particular numbers; the general product law is log(mn) = log m + log n, NOT log(m+n) = log m + log n.

Answer: Both sides equal log 6 (special case only, not a general law)

Q27. If $\log[(x+y)/3] = (1/2)(\log x + \log y)$, prove that $x^2 + y^2 = 7xy$.

[Standard proof-based question]

Solution:

RHS = $(1/2)(\log x + \log y) = (1/2) \log(xy) = \log(\sqrt{(xy)})$.

So $\log[(x+y)/3] = \log(\sqrt{(xy)}) \implies (x+y)/3 = \sqrt{(xy)}$.

Squaring both sides: $(x+y)^2/9 = xy \implies (x+y)^2 = 9xy$.

Expanding: $x^2 + 2xy + y^2 = 9xy \implies x^2 + y^2 = 7xy$. (Proved)

Answer: $x^2 + y^2 = 7xy$ (proved)

Q28. Find the value of $\log_3 81 + \log_2 32 - \log_5 125$.

[Direct evaluation — common short question]

Solution:

$\log_3 81 = \log_3 3^4 = 4$.

$\log_2 32 = \log_2 2^5 = 5$.

$\log_5 125 = \log_5 5^3 = 3$.

So the expression = $4 + 5 - 3 = 6$.

Answer: Value = 6

Q29. If $a^x = b^y = c^z$ and $b^2 = ac$, prove that $y = 2xz/(x+z)$.

[Important long-answer identity question]

Solution:

Let $a^x = b^y = c^z = k$. Then $a = k^{1/x}$, $b = k^{1/y}$, $c = k^{1/z}$.

Given $b^2 = ac$: $k^{2/y} = k^{1/x} \cdot k^{1/z} = k^{1/x + 1/z}$.

Equating exponents: $2/y = 1/x + 1/z = (x+z)/xz$.

So $y = 2xz/(x+z)$. (Proved)

Answer: $y = 2xz/(x+z)$ (proved)

Q30. Evaluate: $\log_{10} 0.001$.

[Basic evaluation with negative index]

Solution:

$0.001 = 1/1000 = 10^{-3}$.

$\log_{10} 0.001 = \log_{10} 10^{-3} = -3 \log_{10} 10 = -3$.

Answer: $\log_{10} 0.001 = -3$

Q31. If $\log_2 x + \log_4 x + \log_{16} x = 21/4$, find x.

[Change-of-base application — important long question]

Solution:

$\log_4 x = \log_2 x / \log_2 4 = \log_2 x / 2$. Similarly $\log_{16} x = \log_2 x / \log_2 16 = \log_2 x / 4$.

So the equation becomes $\log_2 x + (\log_2 x)/2 + (\log_2 x)/4 = 21/4$.

Taking $\log_2 x = t$: $t(1 + 1/2 + 1/4) = 21/4 \implies t(7/4) = 21/4 \implies t = 3$.

So $\log_2 x = 3 \implies x = 2^3 = 8$.

Answer: x = 8
