Chapter: MAXIMA AND MINIMA

1. Introduction

Maxima and Minima is one of the most important applications of derivatives. A function y=f(x) is said to have a LOCAL MAXIMUM at x=a if f(a) is greater than the values of f(x) immediately around it; it has a LOCAL MINIMUM if f(a) is smaller than the values immediately around it. Both are collectively called EXTREME VALUES (or extrema). This chapter uses the first and second derivatives to locate these points precisely, and is heavily applied to real-world optimization problems — maximizing area/profit/volume, or minimizing cost/perimeter/surface area — making it one of the most practically useful and frequently examined topics.

2. Stationary (Critical) Points

A point x=a is called a stationary point (or critical point) of y=f(x) if:

• dy/dx = 0   at x = a

This is the NECESSARY condition for a maximum or minimum to occur at x=a (though not every stationary point is automatically a max or min — it could also be a point of inflection).

3. First Derivative Test

At a stationary point x=a, examine the sign of dy/dx just before and just after x=a:

- If dy/dx changes from POSITIVE to NEGATIVE (as x increases through a): local MAXIMUM at x=a.
- If dy/dx changes from NEGATIVE to POSITIVE: local MINIMUM at x=a.
- If dy/dx does NOT change sign: x=a is neither a maximum nor a minimum (a point of inflection).

4. Second Derivative Test (Faster Method)

At a stationary point x=a (where dy/dx=0):

• If $d^2y/dx^2 < 0$ at x=a ⟹ local MAXIMUM

• If $d^2y/dx^2 > 0$ at x=a ⟹ local MINIMUM

• If $d^2y/dx^2 = 0$ at x=a ⟹ test is INCONCLUSIVE (fall back on the first derivative test)

5. Steps for Solving Optimization (Word) Problems

- Step 1: Identify the quantity to be maximized/minimized, and express it as a function of ONE variable (using any given constraint to eliminate the other variable).
- Step 2: Differentiate this function and set the derivative equal to zero to find the critical point(s).
- Step 3: Apply the second derivative test to confirm whether it is a maximum or minimum.

- Step 4: Substitute back to find the actual maximum/minimum value asked for, along with any other requested quantities (like the other dimension).

6. Important Points for Exams

- Always reduce word problems to a single variable using the given constraint (e.g. fixed perimeter, fixed volume, fixed sum) before differentiating.
- Reject any critical values that are negative, zero, or outside the physically valid range (e.g. a side length cannot be negative or exceed the total available material).
- For 'minimum distance' problems, it is often algebraically simpler to minimize the SQUARE of the distance instead of the distance itself — the minimizing x-value is the same either way.
- Classic 'fixed sum, maximum product' and 'fixed product, minimum sum' problems (like x+1/x) are extremely common — recognise this AM-GM-style pattern quickly.
- For cylinder/box optimization problems, always express the objective function (area or volume) in a SINGLE variable using the volume/area constraint before differentiating.

7. 30 Important Questions & Answers (With Step-by-Step Solutions)

The following questions are compiled based on the WBSCTE Mathematics syllabus for the Maxima and Minima chapter and repeatedly tested question patterns seen in external examination papers. Practising all of these thoroughly gives very strong exam coverage.

Q1. Find the critical point of $y = x^2 - 4x + 3$, and determine whether it is a maximum or minimum.

[Basic — second derivative test — asked almost every year]

Solution:

$dy/dx = 2x - 4$. Set $dy/dx=0$: $2x-4=0 \Rightarrow x=2$.

$d^2y/dx^2 = 2 > 0$, so by the second derivative test, this is a MINIMUM.

Minimum value: $y = 4-8+3 = -1$.

Answer: Minimum at x=2, minimum value = −1

Q2. Find the maximum or minimum value of $y = -x^2 + 6x - 5$.

[Basic — second derivative test — common]

Solution:

$dy/dx = -2x+6$. Set = 0: $x = 3$.

$d^2y/dx^2 = -2 < 0$, so this is a MAXIMUM.

Maximum value: $y = -9+18-5 = 4$.

Answer: Maximum at x=3, maximum value = 4

Q3. Find the local maxima and minima of $y = x^3 - 3x^2 - 9x + 5$.

[Classic important cubic-function question — asked frequently]

Solution:

$dy/dx = 3x^2-6x-9 = 3(x-3)(x+1)$. Set = 0: $x=3$ or $x=-1$.

$d^2y/dx^2 = 6x-6$.

At $x=-1$: $d^2y/dx^2 = -12 < 0 \Rightarrow$ local MAXIMUM; $y=-1-3+9+5=10$.

At $x=3$: $d^2y/dx^2 = 12 > 0 \Rightarrow$ local MINIMUM; $y=27-27-27+5=-22$.

Answer: Local max = 10 at x=−1; Local min = −22 at x=3

Q4. Find the maximum value of $y = 4x - x^2$.

[Basic application]

Solution:

$dy/dx = 4-2x$. Set = 0: $x = 2$.

$d^2y/dx^2 = -2 < 0$, so this is a maximum.

Maximum value: $y = 8-4 = 4$.

Answer: Maximum value = 4 (at x=2)

Q5. Find the local extrema of $y = 2x^3-15x^2+36x+10$.

[Classic important cubic-function question]

Solution:

$dy/dx = 6x^2-30x+36 = 6(x-2)(x-3)$. Set = 0: $x=2$ or $x=3$.

$d^2y/dx^2 = 12x-30$.

At $x=2$: $d^2y/dx^2=-6<0 \Rightarrow$ maximum; $y=16-60+72+10=38$.

At $x=3$: $d^2y/dx^2=6>0 \Rightarrow$ minimum; $y=54-135+108+10=37$.

Answer: Local max = 38 at x=2; Local min = 37 at x=3

Q6. A rectangular garden is to be fenced using 200m of fencing. Find the dimensions that maximize the area.

[Classic important optimization word problem — very common]

Solution:

Let the sides be x and y, so $2x+2y=200 \Rightarrow y=100-x$.

Area $A = xy = x(100-x) = 100x-x^2$.

$dA/dx = 100-2x$. Set=0: $x=50$. $d^2A/dx^2=-2<0 \Rightarrow$ maximum. $y=100-50=50$.

Answer: Dimensions 50m × 50m (a square); maximum area = 2500 sq. m

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Q7. Find two positive numbers whose sum is 20 and whose product is maximum.

[Classic important optimization word problem — very common]

Solution:

Let the numbers be x and $20-x$. Product $P(x) = x(20-x) = 20x-x^2$.

$dP/dx = 20-2x$. Set=0: $x=10$.

$d^2P/dx^2 = -2 < 0 \Rightarrow$ maximum. The two numbers are both 10.

Answer: Numbers = 10 and 10; maximum product = 100

Q8. Find two positive numbers x and y such that $x+y=16$ and x^2y is maximum.

[Classic important optimization word problem — important]

Solution:

$y = 16-x$, so $f(x) = x^2(16-x) = 16x^2-x^3$.

$f'(x) = 32x-3x^2 = x(32-3x)$. Set=0: $x=0$ or $x=32/3$.

$f''(x) = 32-6x$. At $x=32/3$: $f''=32-64=-32<0 \Rightarrow$ maximum. $y=16-32/3=16/3$.

Answer: $x = 32/3$, $y = 16/3$ (gives maximum x^2y)

Q9. Find the dimensions of the rectangle of maximum area that can be inscribed in a circle of radius r.

[Classic important optimization — geometry]

Solution:

Let the rectangle have half-sides x, y with $x^2+y^2=r^2$ (diagonal = diameter).

Maximize $f(x)=x^2(r^2-x^2)=r^2x^2-x^4$ (proportional to Area²). $f'(x)=2r^2x-4x^3=2x(r^2-2x^2)$. Set=0: $x=r/\sqrt{2}$.

By symmetry $y=r/\sqrt{2}$ too, so the rectangle is a SQUARE with side $r\sqrt{2}$.

Maximum area = $4xy = 4\times(r/\sqrt{2})\times(r/\sqrt{2}) = 2r^2$.

Answer: The rectangle is a square of side $r\sqrt{2}$; maximum area = $2r^2$

Q10. An open box is made from a square cardboard of side 12cm by cutting equal squares of side x from each corner and folding up the sides. Find x that maximizes the volume.

[Classic important optimization word problem — very common (box-folding type)]

Solution:

Volume $V(x) = x(12-2x)^2 = 144x-48x^2+4x^3$.

$dV/dx = 144-96x+12x^2 = 12(x-2)(x-6)$. Set=0: x=2 or x=6 (x=6 invalid, makes the base 0).

$d^2V/dx^2 = 24x-96$. At x=2: $-48<0 \Rightarrow$ maximum.

Answer: x = 2cm gives maximum volume = 128 cm³

Q11. Find the maximum value of y = sinx + cosx.

[Trigonometric maxima — important]

Solution:

$dy/dx = \cos x-\sin x$. Set=0: $\tan x=1 \Rightarrow x=\pi/4$.

$d^2y/dx^2 = -\sin x-\cos x$. At $x=\pi/4$: $-\sqrt{2} < 0 \Rightarrow$ maximum.

Maximum value = $\sin(\pi/4)+\cos(\pi/4) = 1/\sqrt{2}+1/\sqrt{2} = \sqrt{2}$.

Answer: Maximum value = $\sqrt{2}$ (at $x=\pi/4$)

Q12. Find the minimum value of f(x) = x + 1/x, for x > 0.

[Classic important optimization (AM-GM type) — very common]

Solution:

$f'(x) = 1-1/x^2$. Set=0: $x^2=1 \Rightarrow x=1$ (taking positive root).

$f''(x) = 2/x^3$. At x=1: $2>0 \Rightarrow$ minimum.

Minimum value: f(1) = 1+1 = 2.

Answer: Minimum value = 2 (at x=1)

Q13. Show that $f(x)=x^3$ has neither a maximum nor a minimum at x=0.

[Classic conceptual edge-case question — important]

Solution:

$f'(x)=3x^2$. Set=0: $x=0$.

$f''(x)=6x$. At $x=0$: $f''(0)=0$, so the second derivative test is inconclusive.

Checking the sign of $f'(x)=3x^2$ around $x=0$: it is positive on BOTH sides (since $x^2 \geq 0$ always). Since the sign of $f'(x)$ does not change, $x=0$ is neither a maximum nor a minimum — it is a point of inflection.

Answer: x=0 is neither a max nor min (point of inflection)

Q14. Find the maximum area of a rectangle with perimeter 40cm.

[Classic important optimization word problem]

Solution:

Let sides x, y with $2x+2y=40 \Rightarrow y=20-x$.

$A(x)=x(20-x)=20x-x^2$. $dA/dx=20-2x$. Set=0: $x=10$.

$d^2A/dx^2=-2<0 \Rightarrow$ maximum. $y=10$ (a square). Maximum area=100 sq cm.

Answer: Square of side 10cm; maximum area = 100 sq. cm

Q15. Find the minimum value of the perimeter of a rectangle whose area is 36 sq. cm.

[Classic important optimization (constrained-area type) — important]

Solution:

Let sides x, y with $xy=36 \Rightarrow y=36/x$.

Perimeter $P(x) = 2x+72/x$. $dP/dx = 2-72/x^2$. Set=0: $x^2=36 \Rightarrow x=6$.

$d^2P/dx^2 = 144/x^3 > 0$ at $x=6 \Rightarrow$ minimum. $y=6$ (a square). Minimum perimeter = 2(6+6) = 24cm.

Answer: Square of side 6cm; minimum perimeter = 24cm

Q16. Find the value of x for which $f(x)=x^2-6x+11$ attains its minimum value, and find that minimum value.

[Basic application]

Solution:

$f'(x) = 2x-6$. Set=0: $x=3$.

$f''(x)=2>0 \Rightarrow$ minimum.

Minimum value: $f(3) = 9-18+11 = 2$.

Answer: Minimum value = 2 (at x=3)

Q17. Find the local maximum value of $f(x) = 12x - x^3$.

[Classic important cubic function question]

Solution:

$f'(x) = 12-3x^2$. Set=0: $x^2=4 \Rightarrow x=\pm 2$.

$f''(x) = -6x$. At $x=2$: $f''=-12<0 \Rightarrow$ maximum; $f(2)=24-8=16$.

At $x=-2$: $f''=12>0 \Rightarrow$ minimum; $f(-2)=-24+8=-16$.

Answer: Local maximum = 16 (at x=2)

Q18. A farmer wants to fence a rectangular field along a straight river (no fencing needed on the river side) using 300m of fencing. Find the dimensions for maximum area.

[Classic important optimization word problem (3-side fencing) — important]

Solution:

Let x = length of each side perpendicular to the river, y = side parallel to the river. Total fencing: $2x+y=300 \Rightarrow y=300-2x$.

Area $A(x) = x(300-2x) = 300x-2x^2$. $dA/dx = 300-4x$. Set=0: $x=75$.

$d^2A/dx^2=-4<0 \Rightarrow$ maximum. $y=300-150=150$. Maximum area = $75\times150 = 11250$ sq. m.

Answer: 75m (perpendicular) × 150m (parallel to river); max area = 11250 sq. m

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Q19. Find the minimum value of $f(x) = 2x^2 - 8x + 15$.

[Basic application]

Solution:

$f'(x) = 4x-8$. Set=0: $x=2$.

$f''(x) = 4 > 0 \Rightarrow$ minimum.

Minimum value: $f(2) = 8-16+15 = 7$.

Answer: Minimum value = 7 (at x=2)

Q20. Show that, among all rectangles with a given fixed area A, the square has the minimum perimeter.

[Classic important general proof — important]

Solution:

Let sides x, y with $xy=A \Rightarrow y=A/x$. Perimeter $P(x)=2x+2A/x$.

$dP/dx = 2-2A/x^2$. Set=0: $x^2=A \Rightarrow x=\sqrt{A}$.

$d^2P/dx^2 = 4A/x^3 > 0$ for $x>0 \Rightarrow$ minimum. $y=A/\sqrt{A}=\sqrt{A}$, so $x=y=\sqrt{A}$ — the rectangle is a square. (Proved)

Answer: The square (side √A) gives minimum perimeter among all fixed-area rectangles (proved)

Q21. Find two positive numbers whose sum is 15 and the sum of whose squares is minimum.

[Classic important optimization word problem]

Solution:

Let numbers be x and $15-x$. $f(x)=x^2+(15-x)^2$.

$f'(x) = 2x-2(15-x) = 4x-30$. Set=0: $x=7.5$.

$f''(x)=4>0 \Rightarrow$ minimum. Both numbers are 7.5.

Answer: Numbers = 7.5 and 7.5

Q22. Find the maximum value of $y = x(10-x)$ for $x \in [0,10]$.

[Basic application]

Solution:

$dy/dx = 10-2x$. Set=0: $x=5$.

$d^2y/dx^2 = -2 < 0 \Rightarrow$ maximum.

Maximum value: $y = 5\times5 = 25$.

Answer: Maximum value = 25 (at x=5)

Q23. An open-top cylindrical can is to have a fixed volume V. Find the ratio of height to radius that minimizes the surface area.

[Classic important optimization (cylinder, open top) — important]

Solution:

Volume: $V=\pi r^2 h \Rightarrow h=V/(\pi r^2)$. Surface area (base + curved surface only): $S=\pi r^2+2\pi rh=\pi r^2+2V/r$.

$dS/dr = 2\pi r-2V/r^2$. Set=0: $\pi r^3=V \Rightarrow r^3=V/\pi$.

Since $V=\pi r^3$: $h=V/(\pi r^2)=\pi r^3/(\pi r^2)=r$. So the ratio h:r = 1:1.

Answer: h : r = 1 : 1 (height equals radius)

Q24. Find the point on the curve $y=x^2$ which is nearest to the point (0,3).

[Classic important distance-minimization problem — important]

Solution:

Let the point on the curve be (x, x^2). Distance²: $D(x)=x^2+(x^2-3)^2=x^4-5x^2+9$.

$D'(x)=4x^3-10x=2x(2x^2-5)$. Set=0: $x=0$ or $x^2=5/2$.

$D''(x)=12x^2-10$. At $x=0$: $-10<0$ (max, reject); at $x^2=5/2$: $20>0 \Rightarrow$ minimum. So $x=\pm\sqrt{5/2}$.

Answer: Nearest points: $(\pm\sqrt{5/2}, 5/2)$

Q25. The profit function of a company is $P(x) = -2x^2+40x-150$, where x is the number of units produced. Find the number of units that maximizes profit, and the maximum profit.

[Classic important economics-application question]

Solution:

$P'(x) = -4x+40$. Set=0: $x=10$.

$P''(x)=-4<0 \Rightarrow$ maximum.

Maximum profit: $P(10) = -200+400-150 = 50$.

Answer: x = 10 units; maximum profit = 50

Q26. Find the local maxima and minima of $f(x) = x^4-4x^3+4x^2+1$.

[Classic important higher-order polynomial question]

Solution:

$f'(x)=4x^3-12x^2+8x=4x(x-1)(x-2)$. Set=0: $x=0,1,2$.

$f''(x)=12x^2-24x+8$. At $x=0$: $8>0$ (min, $f=1$). At $x=1$: $-4<0$ (max, $f=2$). At $x=2$: $8>0$ (min, $f=1$).

So local minima at $x=0$ and $x=2$ (value 1 each), local maximum at $x=1$ (value 2).

Answer: Local minima = 1 (at x=0, x=2); Local maximum = 2 (at x=1)

Q27. A ball thrown vertically upward has height $h(t)=40t-5t^2$ at time t. Find the maximum height reached and the time taken.

[Classic important physics-application question]

Solution:

$h'(t) = 40-10t$. Set=0: $t=4$.

$h''(t) = -10 < 0 \Rightarrow$ maximum.

Maximum height: $h(4) = 160-80 = 80$m.

Answer: Maximum height = 80m, reached at t=4 seconds

Q28. Find the maximum value of $f(x)=3\sin x-4\sin^3 x$ on $[0, \pi/2]$.

[Trigonometric maxima using a triple-angle identity — important]

Solution:

Recall the identity $\sin 3x = 3\sin x-4\sin^3 x$, so $f(x)=\sin 3x$.

$f'(x)=3\cos 3x$. Set=0: $\cos 3x=0 \Rightarrow 3x=\pi/2 \Rightarrow x=\pi/6$.

$f''(x)=-9\sin 3x$. At $x=\pi/6$: $\sin(\pi/2)=1$, so $f''=-9<0 \Rightarrow$ maximum. Maximum value = $\sin(\pi/2)=1$.

Answer: Maximum value = 1 (at x=π/6)

Q29. Find the minimum value of $f(x) = x^2 + 16/x^2$, for $x > 0$.

[Classic important optimization (AM-GM type)]

Solution:

$f'(x) = 2x-32/x^3$. Set=0: $x^4=16 \Rightarrow x=2$ (positive root).

$f''(x) = 2+96/x^4$. At $x=2$: $2+6=8>0 \Rightarrow$ minimum.

Minimum value: $f(2) = 4+4 = 8$.

Answer: Minimum value = 8 (at x=2)

Q30. A closed cylindrical can is to have a fixed volume V. Find the ratio of height to radius that minimizes the total surface area.

[Classic important optimization (closed cylinder) — classic closing question]

Solution:

$S=2\pi r^2+2\pi rh$, with $V=\pi r^2 h \Rightarrow h=V/(\pi r^2)$. So $S(r)=2\pi r^2+2V/r$.

$dS/dr = 4\pi r - 2V/r^2$. Set=0: $2\pi r^3 = V \Rightarrow r^3 = V/(2\pi)$.

Since $V = 2\pi r^3$: $h = V/(\pi r^2) = 2\pi r^3/(\pi r^2) = 2r$. So the ratio h:r = 2:1.

Answer: h : r = 2 : 1 (height equals the diameter)
