Chapter: PROPERTIES OF TRIANGLE

1. Introduction

In any triangle ABC, the sides a, b, c (opposite to angles A, B, C respectively) and the angles are related by several important formulas collectively called the Properties of a Triangle. These include the Sine Rule, the Cosine Rule, the Projection Formula, and formulas for the area of a triangle (including Heron's formula). These formulas allow us to 'solve a triangle' — that is, find all unknown sides and angles — given a suitable combination of known sides and angles (such as SAS, SSS, ASA, or AAS). This topic is extremely important and combines algebra with trigonometry in a highly practical, scoring way.

2. The Sine Rule

For a triangle ABC with circumradius R:

- **$a/\sin A = b/\sin B = c/\sin C = 2R$**

Used mainly when two angles and one side are known (AAS/ASA), or two sides and a non-included angle are known (SSA).

3. The Cosine Rule

- **$\cos A = (b^2+c^2-a^2) / (2bc)$**
- **$\cos B = (a^2+c^2-b^2) / (2ac)$**
- **$\cos C = (a^2+b^2-c^2) / (2ab)$**

Equivalently: $a^2 = b^2+c^2-2bc\cdot\cos A$ (and similarly for b^2, c^2). Used mainly when three sides are known (SSS), or two sides and the included angle are known (SAS).

4. The Projection Formula

- **$a = b\cdot\cos C + c\cdot\cos B$**
- **$b = c\cdot\cos A + a\cdot\cos C$**
- **$c = a\cdot\cos B + b\cdot\cos A$**

Each side equals the sum of the projections of the other two sides onto it.

5. Area of a Triangle

- **Area (Δ) = $(1/2)ab\cdot\sin C = (1/2)bc\cdot\sin A = (1/2)ca\cdot\sin B$**

Heron's Formula (when all three sides are known):

- **$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = (a+b+c)/2$ (semi-perimeter)**

Relation between area, sides, and circumradius:

• **Δ = abc / (4R)**

6. Important Points for Exams

- Use the SINE RULE when you know: two angles + one side (AAS/ASA), or two sides + a non-included angle (SSA).
- Use the COSINE RULE when you know: all three sides (SSS), or two sides + the included angle (SAS).
- The largest angle of a triangle is always opposite the largest side — a useful shortcut when asked to find a specific angle without computing all three.
- If $a^2+b^2=c^2$ for a triangle's sides, the triangle is right-angled at the angle opposite c (converse of Pythagoras) — this often gives a quick shortcut for area (half the product of the two legs).
- Heron's formula is best used when a quick 'nice' factorisation of $s(s-a)(s-b)(s-c)$ is likely, as in classic triangles like 3-4-5, 13-14-15, 5-12-13.

7. 30 Important Questions & Answers (With Step-by-Step Solutions)

The following questions are compiled based on the WBSCTE Mathematics I syllabus for the Properties of Triangle chapter and repeatedly tested question patterns seen in external examination papers. Practising all of these thoroughly gives very strong exam coverage.

Q1. State the Sine Rule for a triangle ABC.

[Core formula — direct recall — asked almost every year]

Solution:

For any triangle ABC with sides a,b,c opposite to angles A,B,C respectively, and circumradius R:

$a/\sin A = b/\sin B = c/\sin C = 2R$.

Answer: $a/\sin A = b/\sin B = c/\sin C = 2R$

Q2. State the Cosine Rule for angle A of a triangle ABC.

[Core formula — direct recall]

Solution:

The Cosine Rule expresses cosA in terms of the three sides:

$\cos A = (b^2+c^2-a^2) / (2bc)$.

Answer: $\cos A = (b^2+c^2-a^2)/(2bc)$

Q3. State the Projection Formula for side a of a triangle ABC.

[Core formula — direct recall]

Solution:

The Projection Formula expresses one side as the sum of projections of the other two:

$a = b\cdot\cos C + c\cdot\cos B$.

Answer: $a = b\cdot\cos C + c\cdot\cos B$

Q4. In triangle ABC, a=7, b=8, and angle C=60°. Find side c using the Cosine Rule.

[Classic important application (SAS case) — very common]

Solution:

$c^2 = a^2+b^2-2ab\cdot\cos C = 49+64-2(7)(8)\cos 60°$.

$= 113 - 112(0.5) = 113 - 56 = 57$.

$c = \sqrt{57}$.

Answer: $c = \sqrt{57}$

Q5. In triangle ABC, if a=3, b=4, c=5, find angle C.

[Classic important application (SSS case) — very common]

Solution:

$\cos C = (a^2+b^2-c^2)/(2ab) = (9+16-25)/(2\times3\times4) = 0/24$.

$= 0$.

So $C = 90°$.

Answer: C = 90°

Q6. In triangle ABC, if a=6, b=8, c=10, find angle A.

[Classic important application (SSS case) — common]

Solution:

$\cos A = (b^2+c^2-a^2)/(2bc) = (64+100-36)/(2\times8\times10) = 128/160$.

$= 0.8$.

$A = \cos^{-1}(0.8) \approx 36.87°$.

Answer: A ≈ 36.87°

Q7. In triangle ABC, angle A=45°, angle B=60°, side a=10. Find side b using the Sine Rule.

[Classic important application (AAS case) — very common]

Solution:

By the Sine Rule: $a/\sin A = b/\sin B$.

$b = a\cdot\sin B/\sin A = 10 \times \sin 60°/\sin 45° = 10 \times (\sqrt{3}/2)/(1/\sqrt{2})$.

$= 10 \times (\sqrt{3}/2) \times \sqrt{2} = 5\sqrt{6}$.

Answer: b = 5√6

Q8. In triangle ABC, a=10, b=12, angle C=50°. Find the area of the triangle.

[Area using (1/2)ab sinC — very important]

Solution:

Area $= (1/2)ab\cdot\sin C = (1/2)(10)(12)\sin 50°$.

$= 60 \times \sin 50° \approx 60 \times 0.766$.

≈ 45.96 sq. units.

Answer: Area ≈ 45.96 sq. units

Q9. In triangle ABC, a=13, b=14, c=15. Find the area using Heron's formula.

[Heron's formula — very important, classic 13-14-15 triangle]

Solution:

$s = (a+b+c)/2 = (13+14+15)/2 = 21$.

Area $= \sqrt{[s(s-a)(s-b)(s-c)]} = \sqrt{[21\times8\times7\times6]}$.

$= \sqrt{7056} = 84$.

Answer: Area = 84 sq. units

Q10. Prove the projection formula: $a = b \cdot \cos C + c \cdot \cos B$.

[Standard proof — very important]

Solution:

Drop a perpendicular from A to BC, meeting BC at D.

In right triangle ABD: $BD = AB \cdot \cos B = c \cdot \cos B$. In right triangle ACD: $DC = AC \cdot \cos C = b \cdot \cos C$.

Since $BD + DC = BC = a$: $a = c \cdot \cos B + b \cdot \cos C$. (Proved)

Answer: $a = b \cdot \cos C + c \cdot \cos B$ (proved)

Q11. In triangle ABC, if a=2, b=3, c=4, verify the projection formula $a = b \cdot \cos C + c \cdot \cos B$.

[Numerical verification of projection formula — important]

Solution:

$\cos B = (a^2+c^2-b^2)/(2ac) = (4+16-9)/16 = 11/16$.

$\cos C = (a^2+b^2-c^2)/(2ab) = (4+9-16)/12 = -3/12 = -1/4$.

$b \cdot \cos C + c \cdot \cos B = 3(-1/4) + 4(11/16) = -3/4 + 11/4 = 8/4 = 2 = a$. (Verified)

Answer: $b \cdot \cos C + c \cdot \cos B = 2 = a$ (verified)

Q12. Find the circumradius R of a triangle with a=6, A=60°.

[Circumradius from Sine Rule — important]

Solution:

Using $a/\sin A = 2R$: $2R = 6/\sin 60° = 6/(\sqrt{3}/2) = 12/\sqrt{3}$.

$2R = 4\sqrt{3}$.

$R = 2\sqrt{3}$.

Answer: $R = 2\sqrt{3}$

Q13. In triangle ABC, angle A=90°, b=3, c=4. Find side a and the area.

[Right triangle special case — common]

Solution:

Since A=90° ($\cos A=0$), the Cosine Rule reduces to Pythagoras: $a^2 = b^2+c^2 = 9+16 = 25$.

$a = 5$.

$\text{Area} = (1/2)bc \cdot \sin A = (1/2)(3)(4)(1) = 6$.

Answer: a = 5, Area = 6 sq. units

Q14. In triangle ABC, a=5, b=7, c=8. Find cosA.

[Basic Cosine Rule application]

Solution:

$\cos A = (b^2+c^2-a^2)/(2bc) = (49+64-25)/(2\times7\times8)$.

$= 88/112$.

$= 11/14$.

Answer: $\cos A = 11/14$

Q15. In triangle ABC, a=5, b=7, c=8. Find the area of the triangle.

[Heron's formula application — common]

Solution:

$s = (5+7+8)/2 = 10$.

Area $= \sqrt{[s(s-a)(s-b)(s-c)]} = \sqrt{[10\times5\times3\times2]} = \sqrt{300}$.

$= 10\sqrt{3}$ sq. units.

Answer: Area = $10\sqrt{3}$ sq. units

Q16. Two sides of a triangle are 8 cm and 5 cm, and the angle between them is 60°. Find the third side.

[Classic important application (SAS case) — common]

Solution:

Using the Cosine Rule: $c^2 = a^2+b^2-2ab\cdot\cos C = 64+25-2(8)(5)\cos 60^\circ$.

$= 89 - 80(0.5) = 89-40 = 49$.

$c = 7$ cm.

Answer: Third side = 7 cm

Q17. Find the area of a triangle with two sides 8 cm and 5 cm, and an included angle of 60° between them.

[Area using (1/2)ab sinC — common]

Solution:

Area $= (1/2)(8)(5)\sin 60^\circ = 20 \times (\sqrt{3}/2)$.

$= 10\sqrt{3}$ sq. cm.

Answer: Area = $10\sqrt{3}$ sq. cm

Q18. In triangle ABC, A=30°, B=45°, and a=10. Find angle C and side c.

[Classic important application (ASA case) — important]

Solution:

$C = 180^\circ-30^\circ-45^\circ = 105^\circ$.

By the Sine Rule: $c = a\cdot\sin C/\sin A = 10\cdot\sin 105^\circ/\sin 30^\circ$.

Using $\sin 105^\circ = (\sqrt{6}+\sqrt{2})/4$: $c = 10 \times [(\sqrt{6}+\sqrt{2})/4] \div (1/2) = 5(\sqrt{6}+\sqrt{2})$.

Answer: $C = 105°$, $c = 5(\sqrt{6}+\sqrt{2})$

Q19. Show that the Cosine Rule $\cos A=(b^2+c^2-a^2)/(2bc)$ can be rearranged to the form $a^2=b^2+c^2-2bc\cdot\cos A$.

[Rearrangement demonstration — basic conceptual question]

Solution:

Starting from $\cos A = (b^2+c^2-a^2)/(2bc)$, multiply both sides by $2bc$:

$2bc\cdot\cos A = b^2+c^2-a^2$.

Rearranging: $a^2 = b^2+c^2-2bc\cdot\cos A$. (Shown — both forms are equivalent)

Answer: $a^2 = b^2+c^2-2bc\cdot\cos A$ (equivalent form shown)

Q20. In a triangle, if $a=9$, $b=10$, $c=11$, find the largest angle.

[Classic important application — largest angle opposite largest side]

Solution:

The largest angle is opposite the largest side, i.e. angle C (opposite $c=11$).

$\cos C = (a^2+b^2-c^2)/(2ab) = (81+100-121)/(2\times9\times10) = 60/180 = 1/3$.

$C = \cos^{-1}(1/3) \approx 70.53°$.

Answer: Largest angle $C \approx 70.53°$

Q21. Find the circumradius R of a triangle with sides $a=13$, $b=14$, $c=15$, given its area is 84.

[Circumradius from area formula $\Delta=abc/4R$ — important]

Solution:

Using $\Delta = abc/(4R)$, we get $R = abc/(4\Delta)$.

$R = (13\times14\times15)/(4\times84) = 2730/336$.

Simplifying (dividing by 42): $R = 65/8 = 8.125$.

Answer: $R = 65/8 = 8.125$

Q22. Show that $a/\sin A = b/\sin B$ in any triangle, using the Sine Rule.

[Basic conceptual demonstration]

Solution:

By the Sine Rule, $a/\sin A = 2R$ and $b/\sin B = 2R$ for the same triangle.

Since both fractions equal the same constant $2R$, they must be equal to each other.

Hence $a/\sin A = b/\sin B$. (Shown)

Answer: $a/\sin A = b/\sin B$ (shown, both equal $2R$)

Q23. In triangle ABC, if angle $A=60°$ and angle $B=45°$, find the ratio $a:b$ using the Sine Rule.

[Ratio application of Sine Rule — common]

Solution:

By the Sine Rule, $a:b = \sin A : \sin B = \sin 60° : \sin 45°$.

$= (\sqrt{3}/2) : (1/\sqrt{2})$.

Multiplying both parts by $2\sqrt{2}$: $= \sqrt{6} : 2$.

Answer: $a : b = \sqrt{6} : 2$

Q24. In triangle ABC, a=5, c=7, and b²=39. Find angle B using the Cosine Rule.

[Reverse Cosine Rule application — important]

Solution:

$b^2 = a^2+c^2-2ac\cdot\cos B$.

$39 = 25+49-2(5)(7)\cos B = 74-70\cos B$.

$70\cos B = 74-39 = 35 \implies \cos B = 0.5$.

$B = 60°$.

Answer: $B = 60°$

Q25. In an equilateral triangle with side a, find the circumradius R.

[Special-case application — common]

Solution:

In an equilateral triangle, every angle = 60°.

Using $a/\sin A = 2R$: $2R = a/\sin 60° = a/(\sqrt{3}/2) = 2a/\sqrt{3}$.

$R = a/\sqrt{3}$.

Answer: $R = a/\sqrt{3}$

Q26. In an equilateral triangle with all sides equal to 10, find the area.

[Special-case application (Heron's formula) — common]

Solution:

$s = (10+10+10)/2 = 15$.

Area $= \sqrt{[15\times5\times5\times5]} = \sqrt{(15\times125)} = \sqrt{1875}$.

Simplify: $\sqrt{1875} = \sqrt{(625\times3)} = 25\sqrt{3}$. (Matches the standard formula Area$=(\sqrt{3}/4)a^2 = (\sqrt{3}/4)(100)=25\sqrt{3}$.)

Answer: Area = $25\sqrt{3}$ sq. units

Q27. Prove the projection formula for side c: a·cosB + b·cosA = c.

[Standard proof — important (analogous to side-a projection formula)]

Solution:

Drop a perpendicular from C to AB, meeting AB at D.

In right triangle ACD: $AD = b\cdot\cos A$. In right triangle BCD: $DB = a\cdot\cos B$.

Since $AD+DB = AB = c$: $b\cdot\cos A + a\cdot\cos B = c$. (Proved)

Answer: $a\cdot\cos B + b\cdot\cos A = c$ (proved)

Q28. In a triangle with a=7, b=24, c=25, show that the triangle is right-angled, and find its area.

[Classic important combined question]

Solution:

Check: $a^2+b^2 = 49+576 = 625 = 25^2 = c^2$. So by the converse of Pythagoras' theorem, the triangle is right-angled at C (angle opposite side c).

Since the right angle is between sides a and b: Area $= (1/2)ab = (1/2)(7)(24)$.

= 84 sq. units.

Answer: Right-angled at C; Area = 84 sq. units

Q29. Find angle C in a triangle where a=1, b=√3, c=2.

[Cosine Rule application — clean standard-angle result]

Solution:

$\cos C = (a^2+b^2-c^2)/(2ab) = (1+3-4)/(2\times1\times\sqrt{3}) = 0/(2\sqrt{3})$.

= 0.

$C = 90°$.

Answer: C = 90°

Q30. Using the Sine Rule, prove that (a−b)/(a+b) = tan[(A−B)/2] / tan[(A+B)/2]. (Law of Tangents)

[Classic important closing identity — combines Sine Rule with sum-to-product formulas]

Solution:

By the Sine Rule, $a=k\cdot\sin A$ and $b=k\cdot\sin B$ for some constant k (=2R). So $(a-b)/(a+b) = (\sin A-\sin B)/(\sin A+\sin B)$.

Using the sum-to-product formulas: $\sin A-\sin B = 2\cos[(A+B)/2]\sin[(A-B)/2]$, $\sin A+\sin B = 2\sin[(A+B)/2]\cos[(A-B)/2]$.

Ratio $= [\cos((A+B)/2)\cdot\sin((A-B)/2)] / [\sin((A+B)/2)\cdot\cos((A-B)/2)] = \tan[(A-B)/2] / \tan[(A+B)/2]$. (Proved)

Answer: $(a-b)/(a+b) = \tan[(A-B)/2]/\tan[(A+B)/2]$ (proved)
