Chapter: 2ND ORDER DERIVATIVES

1. Introduction

The second order derivative of a function $y=f(x)$ is obtained by differentiating the first derivative (dy/dx) once more with respect to x. It is written as d^2y/dx^2 (or $f''(x)$), and represents the rate of change of the slope — geometrically, it tells us about the curvature (concavity) of the graph. A positive d^2y/dx^2 means the curve is concave up; a negative value means concave down; and $d^2y/dx^2=0$ typically identifies a point of inflection. This chapter builds directly on 1st Order Derivatives, and heavily features differential-equation-style 'show that' questions, along with parametric and implicit second derivatives.

2. Definition and Method

The second order derivative is defined as the derivative of the first derivative:

- **$d^2y/dx^2 = d/dx(dy/dx)$**

Method: First find dy/dx using the standard rules of differentiation, then differentiate the resulting expression once more with respect to x, using the same rules (power rule, product rule, quotient rule, chain rule) as needed.

3. Second Derivatives of Standard Functions

- **$y = x^n \implies d^2y/dx^2 = n(n-1)x^{n-2}$**
- **$y = \sin x \implies d^2y/dx^2 = -\sin x \quad (= -y)$**
- **$y = \cos x \implies d^2y/dx^2 = -\cos x \quad (= -y)$**
- **$y = e^x \implies d^2y/dx^2 = e^x$**
- **$y = \ln x \implies d^2y/dx^2 = -1/x^2$**

4. Second Derivative for Parametric Equations

If $x=f(t)$ and $y=g(t)$, first find $dy/dx = (dy/dt)/(dx/dt)$ as usual. Then, to find the second derivative, differentiate this result (dy/dx) with respect to t, and divide by dx/dt once more:

- **$d^2y/dx^2 = [\ d/dt(dy/dx)\ ] / (dx/dt)$**

5. Second Derivative for Implicit Functions

Differentiate the given implicit relation once to find dy/dx in terms of x and y. Then differentiate this expression for dy/dx again with respect to x (applying the Chain Rule to every y-term, and substituting the already-known value of dy/dx where it reappears) to obtain d^2y/dx^2.

6. Important Points for Exams

- A large share of exam questions in this chapter are 'show that' identities (e.g. show $d^2y/dx^2+n^2y=0$) — always simplify fully back to a multiple of y (or the original expression) to complete the proof cleanly.
- For parametric second derivatives, NEVER differentiate dy/dx directly with respect to x — always differentiate with respect to the parameter t first, then divide by dx/dt.
- For implicit second derivatives, it is often easiest to substitute the already-found value of dy/dx back into the expression before simplifying, rather than carrying an unsimplified dy/dx through further algebra.
- Points of inflection (where the curve changes concavity) are found by setting $d^2y/dx^2=0$ and solving for x — this is a key application tested almost every year.
- When a function is built from a product or composition, expect to apply the Product Rule or Chain Rule TWICE in succession — once for dy/dx, and again for d^2y/dx^2.

7. 30 Important Questions & Answers (With Step-by-Step Solutions)

The following questions are compiled based on the WBSCTE Mathematics syllabus for the 2nd Order Derivatives chapter and repeatedly tested question patterns seen in external examination papers. Practising all of these thoroughly gives very strong exam coverage.

Q1. Find d^2y/dx^2 if $y = x^4 - 3x^3 + 2x - 5$.

[Basic — differentiate twice — asked almost every year]

Solution:

First derivative: $dy/dx = 4x^3 - 9x^2 + 2$.

Differentiate again: $d^2y/dx^2 = 12x^2 - 18x$.

Answer: $d^2y/dx^2 = 12x^2 - 18x$

Q2. Find d^2y/dx^2 if $y = x^5 - 4x^3 + 7x$.

[Basic — differentiate twice]

Solution:

First derivative: $dy/dx = 5x^4 - 12x^2 + 7$.

Differentiate again: $d^2y/dx^2 = 20x^3 - 24x$.

Answer: $d^2y/dx^2 = 20x^3 - 24x$

Q3. Find d^2y/dx^2 if $y = \sin x$.

[Basic trigonometric second derivative — very important]

Solution:

First derivative: $dy/dx = \cos x$.

Differentiate again: $d^2y/dx^2 = -\sin x$.

Answer: $d^2y/dx^2 = -\sin x$

Q4. Find d^2y/dx^2 if $y = \cos x$.

[Basic trigonometric second derivative — very important]

Solution:

First derivative: $dy/dx = -\sin x$.

Differentiate again: $d^2y/dx^2 = -\cos x$.

Answer: $d^2y/dx^2 = -\cos x$

Q5. Find d^2y/dx^2 if $y = e^{3x}$.

[Basic exponential second derivative — important]

Solution:

First derivative: $dy/dx = 3e^{3x}$.

Differentiate again: $d^2y/dx^2 = 9e^{3x}$.

Answer: $d^2y/dx^2 = 9e^{3x}$

Q6. Find d^2y/dx^2 if $y = \ln x$.

[Basic logarithmic second derivative — important]

Solution:

First derivative: $dy/dx = 1/x = x^{-1}$.

Differentiate again: $d^2y/dx^2 = -x^{-2} = -1/x^2$.

Answer: $d^2y/dx^2 = -1/x^2$

Q7. Find d^2y/dx^2 if $y = x \cdot e^x$.

[Product rule applied twice — very important]

Solution:

First derivative (Product Rule): $dy/dx = e^x + x \cdot e^x = e^x(1+x)$.

Differentiate again (Product Rule): $d^2y/dx^2 = e^x(1+x) + e^x(1) = e^x(x+2)$.

Answer: $d^2y/dx^2 = e^x(x+2)$

Q8. Find d^2y/dx^2 if $y = x^2 \cdot \sin x$.

[Product rule applied twice (algebraic × trig) — important]

Solution:

First derivative: $dy/dx = 2x \cdot \sin x + x^2 \cdot \cos x$.

Differentiate again: $d^2y/dx^2 = (2\sin x + 2x\cos x) + (2x\cos x - x^2\sin x)$.

$= 2\sin x + 4x\cos x - x^2\sin x$.

Answer: $d^2y/dx^2 = 2\sin x + 4x\cos x - x^2\sin x$

Q9. If $y=\sin x$, show that $d^2y/dx^2 + y = 0$.

[Classic important differential-equation verification]

Solution:

$dy/dx = \cos x$.

$d^2y/dx^2 = -\sin x = -y$.

So $d^2y/dx^2 + y = -y + y = 0$. (Proved)

Answer: $d^2y/dx^2 + y = 0$ (proved)

Q10. If $y=e^{2x}$, show that $d^2y/dx^2 - 4y = 0$.

[Classic important differential-equation verification]

Solution:

$dy/dx = 2e^{2x}$.

$d^2y/dx^2 = 4e^{2x} = 4y$.

So $d^2y/dx^2 - 4y = 4y - 4y = 0$. (Proved)

Answer: $d^2y/dx^2 - 4y = 0$ (proved)

Q11. If $y = A\cdot\cos(nx) + B\cdot\sin(nx)$, show that $d^2y/dx^2 + n^2y = 0$.

[Classic important general identity — frequently asked]

Solution:

$dy/dx = -An\cdot\sin(nx) + Bn\cdot\cos(nx)$.

$d^2y/dx^2 = -An^2\cdot\cos(nx) - Bn^2\cdot\sin(nx) = -n^2[A\cdot\cos(nx)+B\cdot\sin(nx)] = -n^2y$.

So $d^2y/dx^2 + n^2y = -n^2y + n^2y = 0$. (Proved)

Answer: $d^2y/dx^2 + n^2y = 0$ (proved)

Q12. Find d^2y/dx^2 if $y = \tan x$.

[Trigonometric second derivative — common]

Solution:

$dy/dx = \sec^2x$.

Differentiate again (Chain Rule): $d^2y/dx^2 = 2\sec x\cdot(\sec x\cdot\tan x)$.

$= 2\sec^2x\cdot\tan x$.

Answer: $d^2y/dx^2 = 2\sec^2x\cdot\tan x$

Q13. Find d^2y/dx^2 if $y = \ln(\sin x)$.

[Logarithmic-trigonometric second derivative — important]

Solution:

$dy/dx = \cos x/\sin x = \cot x$.

Differentiate again: $d^2y/dx^2 = -\mathrm{cosec}^2x$.

Answer: $d^2y/dx^2 = -\mathrm{cosec}^2x$

Q14. Find the value of d^2y/dx^2 at $x=1$ if $y = x^3-6x^2+9x+2$.

[Second-derivative evaluation at a point — important]

Solution:

$dy/dx = 3x^2-12x+9$.

$d^2y/dx^2 = 6x-12$.

At $x=1$: $d^2y/dx^2 = 6(1)-12 = -6$.

Answer: d^2y/dx^2 at $x=1$ is -6

Q15. Find d^2y/dx^2 if $y = 1/x$.

[Basic — negative power rule applied twice]

Solution:

$dy/dx = -1/x^2 = -x^{-2}$.

$d^2y/dx^2 = 2x^{-3} = 2/x^3$.

Answer: $d^2y/dx^2 = 2/x^3$

Q16. If $x=at^2$, $y=2at$, find d^2y/dx^2 (parametric).

[Parametric second derivative — very important]

Solution:

First: $dy/dt=2a$, $dx/dt=2at$, so $dy/dx = 2a/(2at) = 1/t$.

Differentiate (dy/dx) with respect to t: $d/dt(1/t) = -1/t^2$.

$d^2y/dx^2 = [d/dt(dy/dx)] / (dx/dt) = (-1/t^2)/(2at) = -1/(2at^3)$.

Answer: $d^2y/dx^2 = -1/(2at^3)$

Q17. If $x=a\cdot\cos\theta$, $y=a\cdot\sin\theta$, find d^2y/dx^2 (parametric).

[Parametric second derivative — very important]

Solution:

First: $dy/dx = -\cot\theta$ (found from $dy/d\theta=a\cos\theta$, $dx/d\theta=-a\sin\theta$).

Differentiate (dy/dx) with respect to θ: $d/d\theta(-\cot\theta) = \text{cosec}^2\theta$.

$d^2y/dx^2 = \text{cosec}^2\theta / (dx/d\theta) = \text{cosec}^2\theta / (-a\sin\theta) = -\text{cosec}^3\theta/a$.

Answer: $d^2y/dx^2 = -\text{cosec}^3\theta / a$

Q18. If $x^2+y^2=25$, find d^2y/dx^2 (implicit differentiation).

[Implicit second derivative — classic important question]

Solution:

First derivative: $2x+2y(dy/dx)=0 \Rightarrow dy/dx = -x/y$.

Differentiate again using the Quotient Rule: $d^2y/dx^2 = -[y(1) - x(dy/dx)]/y^2 = -[y - x(-x/y)]/y^2 = -(y^2+x^2)/y^3$.

Since $x^2+y^2=25$: $d^2y/dx^2 = -25/y^3$.

Answer: $d^2y/dx^2 = -25/y^3$

Q19. Find d^2y/dx^2 if $y = \log(x^2+1)$.

[Quotient Rule applied on the first derivative — important]

Solution:

$dy/dx = 2x/(x^2+1)$.

Using the Quotient Rule: $d^2y/dx^2 = [2(x^2+1) - 2x(2x)] / (x^2+1)^2$.

$= (2x^2+2-4x^2)/(x^2+1)^2 = 2(1-x^2)/(x^2+1)^2$.

Answer: $d^2y/dx^2 = 2(1-x^2)/(x^2+1)^2$

Q20. Find the point(s) of inflection of the curve $y = x^3-6x^2+9x+1$ (where $d^2y/dx^2=0$).

[Application of 2nd derivative — very important (basis for curve sketching)]

Solution:

$dy/dx = 3x^2-12x+9$.

$d^2y/dx^2 = 6x-12$.

Set $d^2y/dx^2=0$: $6x-12=0 \Rightarrow x=2$.

Answer: Point of inflection at $x = 2$

Q21. If $y = e^x \cdot \sin x$, find d^2y/dx^2.

[Product Rule twice (exponential × trig) — important]

Solution:

$dy/dx = e^x\sin x + e^x\cos x = e^x(\sin x+\cos x)$.

$d^2y/dx^2 = e^x(\sin x+\cos x) + e^x(\cos x-\sin x) = e^x[(\sin x+\cos x)+(\cos x-\sin x)]$.

$= 2e^x\cos x$.

Answer: $d^2y/dx^2 = 2e^x\cos x$

Q22. If $y = x^n$, find d^2y/dx^2 in terms of n and x.

[General power-rule result — important]

Solution:

$dy/dx = n\cdot x^{n-1}$.

$d^2y/dx^2 = n(n-1)\cdot x^{n-2}$.

Answer: $d^2y/dx^2 = n(n-1)x^{n-2}$

Q23. Find d^2y/dx^2 if $y = (x^2+1)^3$.

[Chain Rule twice — important]

Solution:

$dy/dx = 3(x^2+1)^2\cdot 2x = 6x(x^2+1)^2$.

$d^2y/dx^2 = 6(x^2+1)^2 + 6x\cdot 2(x^2+1)\cdot 2x = 6(x^2+1)^2 + 24x^2(x^2+1)$.

Factorising: $= 6(x^2+1)[(x^2+1)+4x^2] = 6(x^2+1)(5x^2+1)$.

Answer: $d^2y/dx^2 = 6(x^2+1)(5x^2+1)$

Q24. Find d^2y/dx^2 if $y = \sin(2x)$.

[Chain Rule twice (trig with multiple angle) — common]

Solution:

$dy/dx = 2\cos(2x)$.

$d^2y/dx^2 = -4\sin(2x)$.

Answer: $d^2y/dx^2 = -4\sin(2x)$

Q25. Show that if $y=\cos(2x)$, then $d^2y/dx^2 = -4y$.

[Classic important differential-equation verification]

Solution:

$dy/dx = -2\sin(2x)$.

$d^2y/dx^2 = -4\cos(2x) = -4y$. (Proved)

Answer: $d^2y/dx^2 = -4y$ (proved)

Q26. Find d^2y/dx^2 if $y = x\cdot\ln x$.

[Product Rule twice (algebraic × log) — important]

Solution:

$dy/dx = \ln x + x\cdot(1/x) = \ln x + 1$.

$d^2y/dx^2 = 1/x$.

Answer: $d^2y/dx^2 = 1/x$

Q27. If $y = \tan^{-1}x$, find d^2y/dx^2.

[Second derivative of an inverse trig function — important]

Solution:

$dy/dx = 1/(1+x^2) = (1+x^2)^{-1}$.

Differentiate using the Chain Rule: $d^2y/dx^2 = -1\cdot(1+x^2)^{-2}\cdot 2x$.

$= -2x/(1+x^2)^2$.

Answer: $d^2y/dx^2 = -2x/(1+x^2)^2$

Q28. If $y = \sin^{-1}x$, find d^2y/dx^2.

[Second derivative of an inverse trig function — important]

Solution:

$dy/dx = (1-x^2)^{-1/2}$.

Differentiate using the Chain Rule: $d^2y/dx^2 = (-1/2)(1-x^2)^{-3/2}\cdot(-2x) = x(1-x^2)^{-3/2}$.

$= x / (1-x^2)^{3/2}$.

Answer: $d^2y/dx^2 = x/(1-x^2)^{3/2}$

Q29. If $y = A\cdot e^{2x} + B\cdot e^{-2x}$, show that $d^2y/dx^2 = 4y$.

[Classic important general-solution verification]

Solution:

$dy/dx = 2A\cdot e^{2x} - 2B\cdot e^{-2x}$.

$d^2y/dx^2 = 4A\cdot e^{2x} + 4B\cdot e^{-2x} = 4(A\cdot e^{2x}+B\cdot e^{-2x}) = 4y$. (Proved)

Answer: $d^2y/dx^2 = 4y$ (proved)

Q30. Find the value of d^2y/dx^2 at $x=0$ for $y = e^x\cdot \cos x$.

[Second-derivative evaluation at a point — combined application]

Solution:

$dy/dx = e^x\cos x - e^x\sin x = e^x(\cos x-\sin x)$.

$d^2y/dx^2 = e^x(\cos x-\sin x) + e^x(-\sin x-\cos x) = e^x[(\cos x-\sin x)+(-\sin x-\cos x)] = -2e^x\sin x$.

At $x=0$: $d^2y/dx^2 = -2e^0\sin 0 = -2(1)(0) = 0$.

Answer: d^2y/dx^2 at $x=0$ is 0
