WBSCTE — Diploma / Polytechnic

Mathematics — Vector Algebra

Dot Product and Cross Product

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1. Introduction

A vector is a quantity having both magnitude and direction (e.g. force, velocity, displacement), usually written in component form as a = a_{1i} + a_{2j} + a_{3k}, where i, j, k are unit vectors along the x, y, z axes. This chapter covers the two fundamental ways of 'multiplying' two vectors: the Dot Product (Scalar Product), which produces a scalar (number), and the Cross Product (Vector Product), which produces another vector. These operations are essential for finding angles between vectors, checking perpendicularity/parallelism, and computing areas — and are widely used in physics and engineering applications like work, torque, and force.

2. Basic Vector Concepts

- **Magnitude of a = $a_{1i+a2j+a3k}$: |a| = √(a_1^2+a_2^2+a_3^2)**
- **Unit vector along a: â = a / |a|**

3. Dot Product (Scalar Product)

- **Geometric definition: a·b = |a||b|cosθ (θ = angle between a and b)**
- **Component form: a·b = a_{1b1} + a_{2b2} + a_{3b3}**
- **Angle between vectors: cosθ = (a·b) / (|a||b|)**

Key properties:

- a·b = b·a (commutative)
- a·(b+c) = a·b + a·c (distributive)
- a·a = $|a|^2$
- If a·b = 0 (and both are non-zero vectors), then a and b are PERPENDICULAR.
- Projection of a on b = (a·b)/|b|

4. Cross Product (Vector Product)

- **Geometric definition: a×b = |a||b|sinθ · n̂ (n̂ = unit vector perpendicular to both a and b)**
- **Component (determinant) form: a×b = | i j k ; a_1 a_2 a_3 ; b_1 b_2 b_3 |**

Key properties:

- a×b = −(b×a) (anti-commutative — NOT symmetric like the dot product)
- a×a = 0
- If a×b = 0 (and both are non-zero vectors), then a and b are PARALLEL.

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5. Applications of the Cross Product

- **Area of a parallelogram with adjacent sides a, b: Area = |a×b|**
- **Area of a triangle with adjacent sides a, b: Area = (1/2)|a×b|**
- **Scalar Triple Product: a·(b×c) (gives the volume of the parallelepiped formed by a, b, c)**

6. Important Points for Exams

- Dot product gives a SCALAR (a number); cross product gives a VECTOR — never confuse the two, especially in 'find the area' vs 'find the angle' questions.
- For perpendicularity, use the dot product (set a·b=0); for parallelism, use the cross product (set a×b=0) or check that one vector is a scalar multiple of the other.
- Always double-check the sign pattern (+,−,+) when expanding the determinant for a×b — the j-component always carries a negative sign.
- For a triangle given by 3 vertices (not vectors from the origin), first form two side vectors (e.g. AB and AC) by subtracting position vectors, then apply the cross-product area formula.
- Vector identities like $|a+b|^2 = |a|^2+|b|^2+2(a\cdot b)$ are extremely useful shortcuts — memorise this expansion pattern (it mirrors ordinary algebra with a·b playing the role of a cross term).

7. 30 Important Questions & Answers (With Step-by-Step Solutions)

The following questions are compiled based on the WBSCTE Mathematics syllabus for the Vector Algebra (Dot Product and Cross Product) chapter and repeatedly tested question patterns seen in external examination papers. Practising all of these thoroughly gives very strong exam coverage.

Q1. If a = 2i+3j−k and b = i−2j+4k, find a·b.

[Basic dot product calculation — asked almost every year]

Solution:

Dot product formula: $a \cdot b = a_{1b1} + a_{2b2} + a_{3b3}$.

a·b = (2)(1) + (3)(−2) + (−1)(4) = 2 − 6 − 4.

= −8.

Answer: a·b = −8

Q2. If a = i+2j+3k and b = 3i−j+2k, find a·b.

[Basic dot product calculation]

Solution:

a·b = (1)(3) + (2)(−1) + (3)(2).

= 3 − 2 + 6.

= 7.

Answer: a·b = 7

Q3. Find the magnitude of the vector a = 3i−4j+12k.

[Basic magnitude calculation — very important]

Solution:

$|a| = \sqrt{(a_1^2+a_2^2+a_3^2)}$.

$|a| = \sqrt{(3^2+(-4)^2+12^2)} = \sqrt{(9+16+144)} = \sqrt{169}$.

= 13.

Answer: |a| = 13

Q4. Find the angle between the vectors a=i+j and b=j+k.

[Angle between vectors using dot product — very important]

Solution:

a·b = (1)(0)+(1)(1)+(0)(1) = 1. $|a|=\sqrt{2}$, $|b|=\sqrt{2}$.

$\cos\theta = (a \cdot b)/(|a||b|) = 1/(\sqrt{2}\times\sqrt{2}) = 1/2$.

θ = 60°.

Answer: θ = 60°

Q5. Show that a=i+2j+3k and b=3i−3j+k are perpendicular.

[Perpendicularity check via dot product — very important]

Solution:

a·b = (1)(3) + (2)(−3) + (3)(1) = 3−6+3.

= 0.

Since a·b = 0, the vectors are perpendicular. (Proved)

Answer: a·b = 0, hence a ⊥ b (proved)

Q6. Find λ such that a=2i+λj+k and b=i−2j+3k are perpendicular.

[Classic important application question]

Solution:

For perpendicular vectors, a·b = 0.

a·b = (2)(1) + λ(−2) + (1)(3) = 5−2λ.

Set 5−2λ=0 ⟹ λ = 5/2.

Answer: λ = 5/2

Q7. Find the projection of a=2i+3j+k onto b=i+2j+2k.

[Projection formula application — important]

Solution:

Projection of a on b = (a·b)/|b|.

a·b = 2+6+2 = 10. |b| = √(1+4+4) = 3.

Projection = 10/3.

Answer: Projection = 10/3

Q8. Find a×b where a=2i+j−k and b=i−3j+2k.

[Basic cross product (determinant method) — very important]

Solution:

a×b = | i j k ; 2 1 −1 ; 1 −3 2 |.

i-component: (1)(2)−(−1)(−3) = 2−3 = −1. j-component: −[(2)(2)−(−1)(1)] = −5. k-component: (2)(−3)−(1)(1) = −7.

a×b = −i −5j −7k.

Answer: a×b = −i − 5j − 7k

Q9. Find a×b where a=i+j+k and b=2i−j+3k.

[Basic cross product (determinant method) — very important]

Solution:

$a \times b = | i\ j\ k ; 1\ 1\ 1 ; 2\ -1\ 3 |$.

i-component: $(1)(3)-(1)(-1) = 4$. j-component: $-[(1)(3)-(1)(2)] = -1$. k-component: $(1)(-1)-(1)(2) = -3$.

$a \times b = 4i - j - 3k$.

Answer: $a \times b = 4i - j - 3k$

Q10. Find the area of the triangle with adjacent sides $a=2i+3j$ and $b=i-4j$.

[Area of triangle using cross product — very important]

Solution:

$a \times b = | i\ j\ k ; 2\ 3\ 0 ; 1\ -4\ 0 | = k[(2)(-4)-(3)(1)] = -11k$.

$|a \times b| = 11$.

Area of triangle = $(1/2)|a \times b| = 5.5$ sq. units.

Answer: Area = 5.5 sq. units

Q11. Find the area of the parallelogram with adjacent sides $a=3i-j+2k$ and $b=i+3j-k$.

[Area of parallelogram using cross product — very important]

Solution:

$a \times b$: i-component: $(-1)(-1)-(2)(3) = -5$. j-component: $-[(3)(-1)-(2)(1)] = 5$. k-component: $(3)(3)-(-1)(1) = 10$.

$a \times b = -5i+5j+10k$.

$|a \times b| = \sqrt{25+25+100} = \sqrt{150} = 5\sqrt{6}$. Area of parallelogram = $5\sqrt{6}$ sq. units.

Answer: Area = $5\sqrt{6}$ sq. units

Q12. Show that $a=i+2j-k$ and $b=2i+4j-2k$ are parallel.

[Parallel-vector verification (cross product) — important]

Solution:

Notice $b = 2a$, since $2(i+2j-k) = 2i+4j-2k = b$.

Verify via cross product: $a \times b$ component-wise all give 0 (since b is a scalar multiple of a).

Since $a \times b = 0$, the vectors are parallel. (Proved)

Answer: $a \times b = 0$, hence $a \parallel b$ (proved, since $b=2a$)

Q13. Find a unit vector in the direction of $a=3i-4j+12k$.

[Unit vector — basic application]

Solution:

$|a| = 13$ (computed earlier).

Unit vector = $a/|a|$.

$= (3/13)i - (4/13)j + (12/13)k$.

Answer: $\hat{a} = (3/13)i - (4/13)j + (12/13)k$

Q14. If a=2i−j+2k, find |a| and hence a unit vector parallel to a.

[Unit vector — basic application]

Solution:

|a| = √(4+1+4) = √9 = 3.

Unit vector = a/|a|.

= (2/3)i − (1/3)j + (2/3)k.

Answer: |a|=3, â = (2/3)i − (1/3)j + (2/3)k

Q15. Find λ so that a=λi+2j+k and b=4i+4j+2k are parallel.

[Parallel-vector application (ratio method) — important]

Solution:

For parallel vectors, corresponding components must be in the same ratio: λ/4 = 2/4 = 1/2.

From 2/4 = 1/2: this ratio checks out; from 1/2 = 1/2: also consistent.

So λ/4 = 1/2 $\Longrightarrow$ λ = 2.

Answer: λ = 2

Q16. Prove that a·(b+c) = a·b + a·c, using components.

[Standard proof — distributive property of dot product]

Solution:

Let $a=(a_1,a_2,a_3)$, $b=(b_1,b_2,b_3)$, $c=(c_1,c_2,c_3)$.

$a\cdot(b+c) = a_1(b_{1+c1})+a_2(b_{2+c2})+a_3(b_{3+c3}) = (a_{1b1+a2b2+a3b3}) + (a_{1c1+a2c2+a3c3})$.

= a·b + a·c. (Proved)

Answer: a·(b+c) = a·b + a·c (proved)

Q17. If |a|=3, |b|=4, and the angle between them is 60°, find a·b.

[Direct dot-product formula application — common]

Solution:

a·b = |a||b|cosθ.

= 3 × 4 × cos60° = 12 × 0.5.

= 6.

Answer: a·b = 6

Q18. If |a|=5, |b|=7, and the angle between them is 90°, find |a×b|.

[Direct cross-product magnitude formula application — common]

Solution:

$|a \times b| = |a||b|\sin\theta$.

$= 5 \times 7 \times \sin 90° = 35 \times 1$.

$= 35$.

Answer: $|a \times b| = 35$

Q19. If a and b are unit vectors with an angle of 120° between them, find |a+b|.

[Classic important combined-magnitude question]

Solution:

$|a+b|^2 = |a|^2+|b|^2+2(a\cdot b) = 1+1+2(1)(1)\cos 120°$.

$= 2 + 2(-1/2) = 2-1$.

$|a+b| = \sqrt{1} = 1$.

Answer: $|a+b| = 1$

Q20. Find the value of a·a for a=i−2j+3k.

[Basic — relation $a\cdot a=|a|^2$]

Solution:

$a\cdot a = 1^2+(-2)^2+3^2 = 1+4+9$.

$= 14$ (this equals $|a|^2$).

Answer: $a\cdot a = 14$

Q21. Show that a×a = 0 for any vector a.

[Standard conceptual proof — property of cross product]

Solution:

The angle between a vector and itself is 0°.

$|a \times a| = |a||a|\sin 0° = |a|^2 \times 0 = 0$.

Hence a×a is the zero vector. (Proved)

Answer: a×a = 0 (proved)

Q22. If a·b=0 and neither a nor b is the zero vector, what can be concluded about the angle between them?

[Conceptual question — important]

Solution:

$a\cdot b = |a||b|\cos\theta = 0$. Since $|a|\neq 0$ and $|b|\neq 0$, we must have $\cos\theta=0$.

This means $\theta = 90°$.

Conclusion: the vectors a and b are perpendicular to each other.

Answer: a and b are perpendicular ($\theta=90°$)

Q23. Find the angle between a=i+j+k and b=i+j−k.

[Angle between vectors — common]

Solution:

$a \cdot b = 1+1-1 = 1$. $|a|=\sqrt{3}$, $|b|=\sqrt{3}$.

$\cos\theta = 1/(\sqrt{3}\times\sqrt{3}) = 1/3$.

$\theta = \cos^{-1}(1/3)$.

Answer: $\theta = \cos^{-1}(1/3)$

Q24. Find |a−b| if |a|=3, |b|=4, and a·b=6.

[Combined magnitude application — important]

Solution:

$|a-b|^2 = |a|^2+|b|^2-2(a\cdot b) = 9+16-12$.

$= 13$.

$|a-b| = \sqrt{13}$.

Answer: $|a-b| = \sqrt{13}$

Q25. Simplify (a+b)·(a−b) in terms of |a| and |b|.

[General identity — classic conceptual question]

Solution:

$(a+b)\cdot(a-b) = a\cdot a - a\cdot b + b\cdot a - b\cdot b$.

Since $a\cdot b = b\cdot a$, these cross terms cancel: $= a\cdot a - b\cdot b$.

$= |a|^2 - |b|^2$.

Answer: $(a+b)\cdot(a-b) = |a|^2 - |b|^2$

Q26. Two vectors a and b are such that |a|=2, |b|=3, and a·b=−3. Find the angle between them.

[Reverse application of dot-product formula — important]

Solution:

$\cos\theta = (a\cdot b)/(|a||b|) = -3/(2\times3) = -1/2$.

$\theta = 120°$.

Answer: $\theta = 120°$

Q27. If a=2i+3j+k, b=i−2j+k, c=3i+j−2k, find a·(b×c) [scalar triple product].

[Scalar triple product — classic important application]

Solution:

First find b×c: i-component: $(-2)(-2)-(1)(1)=3$. j-component: $-[(1)(-2)-(1)(3)]=5$. k-component: $(1)(1)-(-2)(3)=7$.

b×c = 3i+5j+7k.

$a\cdot(b\times c) = (2)(3)+(3)(5)+(1)(7) = 6+15+7 = 28$.

Answer: a·(b×c) = 28

Q28. Find the area of the triangle with vertices A(1,1,1), B(2,3,4), C(4,3,2) (using position vectors).

[Classic important application combining position vectors + cross product]

Solution:

AB = B−A = (1,2,3). AC = C−A = (3,2,1).

AB×AC: i-component: $(2)(1)-(3)(2)=-4$. j-component: $-[(1)(1)-(3)(3)]=8$. k-component: $(1)(2)-(2)(3)=-4$.

AB×AC = −4i+8j−4k. $|AB\times AC| = \sqrt{16+64+16} = \sqrt{96} = 4\sqrt{6}$. Area = $(1/2)(4\sqrt{6}) = 2\sqrt{6}$ sq. units.

Answer: Area = 2√6 sq. units

Q29. If a+b+c=0, and |a|=3, |b|=5, |c|=7, find the angle between a and b.

[Classic important closing question — vector identity application]

Solution:

Since $c=-(a+b)$: $|c|^2 = |a+b|^2 = |a|^2+|b|^2+2(a\cdot b)$.

$49 = 9+25+2(a\cdot b) \implies 2(a\cdot b) = 15 \implies a\cdot b = 7.5$.

$\cos\theta = (a\cdot b)/(|a||b|) = 7.5/15 = 0.5 \implies \theta = 60°$.

Answer: θ = 60°

Q30. Find the value of m for which a=mi+2j+k and b=4i−2j−2k are perpendicular.

[Classic important closing application]

Solution:

$a\cdot b = 4m + (2)(-2) + (1)(-2) = 4m-4-2 = 4m-6$.

Set $a\cdot b=0$: $4m-6=0$.

$m = 3/2$.

Answer: m = 3/2
