Applied Physics 1

Dimensions and Dimensional Analysis

1. Dimensions and Dimensional Formulae

The dimensions of a physical quantity are the powers to which the fundamental quantities (mass, length, time, electric current, temperature, amount of substance, and luminous intensity) must be raised in order to represent that physical quantity. Dimensions tell us how a derived quantity is built up from the base quantities, without reference to its numerical magnitude or the particular unit used to measure it.

The dimensional formula of a physical quantity is an expression that shows how, and to what power, the fundamental quantities occur in it. It is written by enclosing the powers of M (mass), L (length), and T (time), along with other base symbols where required, in square brackets.

The symbols used for the fundamental quantities in dimensional formulae are:

Fundamental Quantity	**Symbol used in Dimensional Formula**
Length	L
Mass	M
Time	T
Electric Current	I (or A)
Thermodynamic Temperature	θ (or K)
Amount of Substance	N
Luminous Intensity	J (or cd)

For example, since velocity is defined as displacement divided by time, its dimensional formula is obtained as follows:

- Velocity = Displacement / Time = L / T = $[M^0L^1T^{-1}]$

Similarly, the dimensional formulae of several common physical quantities are tabulated below:

Physical Quantity	**Formula**	**Dimensional Formula**	**SI Unit**
Area	Length × Breadth	$[M^0L^2T^0]$	m^2

Volume	Length × Breadth × Height	$[M^0L^3T^0]$	m^3
Velocity	Displacement / Time	$[M^0L^1T^{-1}]$	m/s
Acceleration	Velocity / Time	$[M^0L^1T^{-2}]$	m/s^2
Force	Mass × Acceleration	$[M^1L^1T^{-2}]$	N
Work / Energy	Force × Displacement	$[M^1L^2T^{-2}]$	J
Power	Work / Time	$[M^1L^2T^{-3}]$	W
Density	Mass / Volume	$[M^1L^{-3}T^0]$	kg/m^3
Pressure	Force / Area	$[M^1L^{-1}T^{-2}]$	Pa
Momentum	Mass × Velocity	$[M^1L^1T^{-1}]$	kg·m/s

A dimensional equation is simply an equation obtained by equating a physical quantity with its corresponding dimensional formula. For example, the dimensional equation for force is written as $[F] = [M^1L^1T^{-2}]$.

2. Principle of Homogeneity of Dimensions

The principle of homogeneity of dimensions states that in any physically meaningful equation, the dimensions of every term on both sides of the equation must be identical. In other words, only quantities having the same dimensions can be added, subtracted, or equated to one another.

This principle follows from the simple fact that physical quantities of different natures (for example, length and time) cannot be meaningfully combined by addition or subtraction. Consider the equation of motion:

- $v = u + at$

Here, v (final velocity), u (initial velocity), and at (acceleration × time) must all have the same dimensions, namely $[M^0L^1T^{-1}]$, for the equation to be dimensionally valid. Checking this:

- $[v] = [M^0L^1T^{-1}]$
- $[u] = [M^0L^1T^{-1}]$
- $[at] = [M^0L^1T^{-2}] \times [M^0L^0T^1] = [M^0L^1T^{-1}]$

Since all three terms have the same dimensions, the equation is dimensionally homogeneous and therefore consistent.

Applications of the principle of homogeneity:

1. It is used to check whether a given physical equation is dimensionally correct.
2. It is used to derive relationships between different physical quantities.
3. It is used to convert the value of a physical quantity from one system of units to another.
4. It helps in finding the dimensions of unknown constants appearing in a physical relation.

3. Dimensional Equations and Their Applications

3.1 Conversion from One System of Units to Another

Since a physical quantity has a fixed dimensional formula regardless of the unit system used, dimensional analysis provides a convenient method for converting the numerical value of a quantity from one system of units to another. If a quantity Q has the dimensional formula $[M^aL^bT^c]$, and its magnitude is n_1 units in one system (with fundamental units M_1, L_1, T_1) and n_2 units in another system (with fundamental units M_2, L_2, T_2), then the two measurements are related by:

- $n_1[M_1^aL_1^bT_1^c] = n_2[M_2^aL_2^bT_2^c]$

This relation allows the numerical value of a physical quantity to be converted from, say, the CGS system to the SI system, simply by substituting the ratios of the respective base units raised to the appropriate powers. For example, converting 1 newton (SI unit of force) into dynes (CGS unit of force) using the dimensional formula $[M^1L^1T^{-2}]$ gives 1 N = 10^5 dyne.

3.2 Checking the Dimensional Correctness of Equations

Dimensional analysis is widely used to verify whether a physical equation is dimensionally correct, using the principle of homogeneity. If the dimensions of the terms on the left-hand side and the right-hand side of an equation match, the equation is said to be dimensionally consistent. This method is a quick and reliable check, although, as discussed later, it does not guarantee that the equation is numerically or physically complete.

3.3 Derivation of Simple Equations

Dimensional analysis can also be used to derive the form of simple physical relationships, provided the physical quantities on which a given quantity depends are already known. The general procedure is as follows:

5. Identify the physical quantities on which the required quantity depends.

6. Assume the relation as a product of these quantities, each raised to an unknown power.
7. Write the dimensional formula of each quantity and equate the powers of M, L, and T on both sides.
8. Solve the resulting simultaneous equations to find the unknown powers.
9. Substitute the powers back into the assumed relation to obtain the required formula, apart from a dimensionless constant.

As an illustration, the time period T of a simple pendulum can be shown, using this method, to depend on the length l of the pendulum and the acceleration due to gravity g in the form $T = k\sqrt{(l/g)}$, where k is a dimensionless constant that cannot be determined by dimensional analysis alone.

4. Limitations of Dimensional Analysis

Although dimensional analysis is a powerful and quick technique, it has several important limitations that must be kept in mind:

10. It does not give any information about the dimensionless proportionality constants appearing in a formula; these must be determined experimentally or theoretically.
11. It cannot be applied to equations involving trigonometric, logarithmic, or exponential functions, since these functions themselves must be dimensionless.
12. It cannot be used to derive relations that involve more than three physical quantities having independent dimensions, since only three equations (for M, L, T) are available to solve for the unknown powers.
13. It cannot distinguish between physical quantities that have the same dimensional formula, such as work, energy, and torque, which are all $[M^1L^2T^{-2}]$.
14. It cannot be used to derive a relation if the exact physical quantities on which a quantity depends are not known in advance.
15. A dimensionally correct equation is not necessarily numerically or physically correct, since dimensional consistency alone does not verify the accuracy of a formula.

Because of these limitations, dimensional analysis is generally used as a supportive tool alongside experimental and theoretical methods, rather than as a complete substitute for them.

5. Multiple Choice Questions (MCQs)

Answer the following questions by selecting the most appropriate option.

1. The dimensional formula of force is:

A) $[M^1L^1T^{-1}]$
B) $[M^1L^1T^{-2}]$
C) $[M^1L^2T^{-2}]$
D) $[M^\circ L^1T^{-2}]$

2. Which of the following pairs has the same dimensions?

A) Work and Power
B) Work and Energy
C) Force and Pressure
D) Momentum and Force

3. The dimensional formula of pressure is:

A) $[M^1L^{-1}T^{-2}]$
B) $[M^1L^2T^{-2}]$
C) $[M^1L^1T^{-1}]$
D) $[M^\circ L^\circ T^\circ]$

4. The principle of homogeneity of dimensions states that:

A) Only the LHS of an equation has dimensions
B) All terms on both sides of an equation must have the same dimensions
C) Dimensions apply only to fundamental quantities
D) Dimensions change with the system of units used

5. Which of the following is a dimensionless quantity?

A) Velocity
B) Strain
C) Force
D) Density

6. Dimensional analysis CANNOT be used to determine:

A) Dimensional consistency of an equation
B) The exact numerical constant in a formula
C) Conversion of units between systems
D) The relationship between derived quantities

7. The dimensional formula of energy is the same as that of:

A) Power
B) Torque
C) Momentum
D) Pressure

8. Which method is used to convert a physical quantity from one system of units to another?

A) Method of dimensions

B) Method of interpolation

C) Method of extrapolation

D) Method of significant figures

9. The dimensional formula of velocity is:

A) $[M^0L^1T^{-1}]$

B) $[M^0L^1T^{-2}]$

C) $[M^1L^1T^{-1}]$

D) $[M^0L^2T^{-1}]$

10. An equation that is dimensionally correct is:

A) Always numerically correct

B) Never numerically correct

C) Not necessarily numerically correct

D) Always incorrect

11. The dimensional formula for power is:

A) $[M^1L^2T^{-2}]$

B) $[M^1L^2T^{-3}]$

C) $[M^1L^1T^{-3}]$

D) $[M^1L^3T^{-2}]$

12. Which of the following cannot be determined using dimensional analysis?

A) Correctness of a physical equation

B) Relation among physical quantities

C) Proportionality constants that are dimensionless

D) Units of a derived quantity

13. The dimensional formula of momentum is:

A) $[M^1L^1T^{-1}]$

B) $[M^1L^2T^{-1}]$

C) $[M^1L^1T^{-2}]$

D) $[M^{\circ}L^1T^{-1}]$

14. A dimensional equation is obtained by equating a physical quantity with its:

A) Numerical value

B) Dimensional formula

C) SI unit only

D) CGS unit only

15. Dimensional analysis fails for equations containing:

A) Trigonometric and exponential functions

B) Only fundamental quantities
C) Only derived quantities
D) Constants of proportionality

16. The dimensional formula of density is:
A) $[M^1L^3T^0]$
B) $[M^1L^{-3}T^0]$
C) $[M^{-1}L^3T^0]$
D) $[M^{\circ}L^{-3}T^0]$

Answer Key

1. B) $[M^1L^1T^{-2}]$ | 2. B) Work and Energy | 3. A) $[M^1L^{-1}T^{-2}]$ | 4. B) All terms on both sides of an equation must have the same dimensions | 5. B) Strain | 6. B) The exact numerical constant in a formula | 7. B) Torque | 8. A) Method of dimensions | 9. A) $[M^0L^1T^{-1}]$ | 10. C) Not necessarily numerically correct | 11. B) $[M^1L^2T^{-3}]$ | 12. C) Proportionality constants that are dimensionless | 13. A) $[M^1L^1T^{-1}]$ | 14. B) Dimensional formula | 15. A) Trigonometric and exponential functions | 16. B) $[M^1L^{-3}T^0]$

6. Broad / Long Answer Questions

1. Define dimensions and dimensional formula of a physical quantity. Explain with suitable examples.

2. State and explain the principle of homogeneity of dimensions with a suitable example.

3. Derive the dimensional formula of force, work, and power starting from their basic definitions.

4. What is a dimensional equation? Explain its role in checking the correctness of a physical relation.

5. Explain, with the help of an example, how dimensional analysis is used to convert a physical quantity from one system of units to another.

6. Using the principle of homogeneity, check the dimensional correctness of the equation $v = u + at$.

7. Derive the relation for the time period of a simple pendulum using the method of dimensions.

8. Explain any five limitations of dimensional analysis with suitable examples.

9. Distinguish between fundamental and derived dimensions, giving examples of each.

10. Explain how dimensional analysis helps in deriving simple physical relations, and discuss the conditions under which this method fails.