Applied Physics 1

Moment of Inertia and Radius of Gyration

1. Moment of Inertia and Its Physical Significance

Moment of inertia is a property of a rigid body that measures its resistance to a change in its state of rotational motion about a given axis. It is the rotational analogue of mass in translational motion. Just as mass measures how difficult it is to change the linear velocity of a body (its inertia in translational motion), moment of inertia measures how difficult it is to change the angular velocity of a body (its inertia in rotational motion).

For a system of particles, the moment of inertia I about a given axis is defined as the sum of the products of the mass of each particle and the square of its perpendicular distance from the axis of rotation:

- $I = \Sigma m_i r_i^2$

where m_i is the mass of the i-th particle and r_i is its perpendicular distance from the axis of rotation. The SI unit of moment of inertia is kilogram-metre squared ($kg \cdot m^2$).

1.1 Physical Significance of Moment of Inertia

The physical significance of moment of inertia can be understood through the following points:

- It represents the rotational inertia of a body — the greater the moment of inertia, the greater the torque required to produce a given angular acceleration, in accordance with the relation $\tau = I\alpha$.
- Unlike mass, moment of inertia is not a fixed property of a body alone; it depends on both the total mass of the body and how that mass is distributed relative to the chosen axis of rotation. The same body can have different moments of inertia about different axes.
- For a given mass, a body has a larger moment of inertia when more of its mass is concentrated farther away from the axis of rotation, and a smaller moment of inertia when its mass is concentrated closer to the axis.
- Moment of inertia plays the same role in the rotational equation $\tau = I\alpha$ that mass plays in the translational equation $F = ma$, and it also appears in the expressions for angular momentum ($L = I\omega$) and rotational kinetic energy ($K.E. = \frac{1}{2} I\omega^2$).

- A body with a large moment of inertia is harder to start rotating and harder to stop once rotating, which is why heavy flywheels with a large moment of inertia are used in machines to maintain smooth and steady rotational motion.

2. Radius of Gyration of a Rigid Body

The radius of gyration of a rigid body about a given axis is defined as the distance from the axis of rotation at which, if the entire mass of the body were assumed to be concentrated as a point mass, the moment of inertia of this point mass about the axis would be equal to the actual moment of inertia of the body about that same axis.

If M is the total mass of the body and K is its radius of gyration about a given axis, then the moment of inertia I of the body about that axis is related to K by:

- $I = M K^2$

or equivalently,

- $K = \sqrt{I / M}$

The radius of gyration is a convenient way of expressing the moment of inertia of a body in terms of a single length, making it easier to compare the rotational inertia of bodies of different shapes and mass distributions, even when they have the same total mass. The radius of gyration depends on the shape of the body, the distribution of its mass, and the position of the axis of rotation.

3. Theorems of Parallel and Perpendicular Axes

3.1 Theorem of Parallel Axes

The theorem of parallel axes states that the moment of inertia of a rigid body about any axis is equal to the sum of its moment of inertia about a parallel axis passing through its centre of mass, and the product of the mass of the body and the square of the perpendicular distance between the two parallel axes.

This is expressed as:

- $I = I_{CM} + M d^2$

where I_{CM} is the moment of inertia about an axis through the centre of mass, M is the total mass of the body, and d is the perpendicular distance between the given axis and the axis through the centre of mass. This theorem applies to a rigid body of any shape.

3.2 Theorem of Perpendicular Axes

The theorem of perpendicular axes states that, for a plane laminar (flat, two-dimensional) body, the moment of inertia about an axis perpendicular to the plane of the body is equal to the sum of its moments of inertia about any two mutually perpendicular axes lying in the plane of the body and intersecting each other at the point where the perpendicular axis passes through.

This is expressed as:

- $Iz = I_x + I_y$

where I_x and I_y are the moments of inertia about two mutually perpendicular axes (x and y) lying in the plane of the lamina, and Iz is the moment of inertia about the z-axis, perpendicular to the plane of the lamina, passing through the point of intersection of the x and y axes. This theorem is applicable only to flat, plane laminar bodies, and not to three-dimensional bodies in general.

4. Moment of Inertia of Common Rigid Bodies (Formulae Only)

The moment of inertia of several common rigid bodies of regular shape, about specific standard axes, is given below. These formulae are widely used in engineering calculations involving rotating machine components such as shafts, flywheels, and discs.

Rigid Body	**Axis of Rotation**	**Moment of Inertia (I)**
Thin uniform rod	Passing through the centre, perpendicular to the length	$I = ML^2/12$
Thin uniform rod	Passing through one end, perpendicular to the length	$I = ML^2/3$
Circular disc (solid)	Passing through the centre, perpendicular to the plane	$I = MR^2/2$
Circular disc (solid)	Diameter (in the plane of the disc)	$I = MR^2/4$
Circular ring (hoop)	Passing through the centre, perpendicular to the plane	$I = MR^2$
Circular ring (hoop)	Diameter (in the plane of the ring)	$I = MR^2/2$
Solid sphere	Passing through the centre (diameter)	$I = 2MR^2/5$
Hollow sphere (thin spherical shell)	Passing through the centre (diameter)	$I = 2MR^2/3$

In each formula, M represents the total mass of the body, L represents the length (for a rod), and R represents the radius (for a disc, ring, or sphere). These results show that, for the same mass and radius, a ring has a greater moment of inertia than a disc about a central axis perpendicular to its plane, since the mass of a ring is concentrated entirely at the rim, farther from the axis, whereas the mass of a disc is distributed throughout its area. Similarly, a hollow sphere has a greater moment of inertia than a solid sphere of the same mass and radius, since all the mass of the hollow sphere lies at the outer surface, farther from the central axis.

5. Simple Numerical Problems

Worked Example 1:
A uniform rod of mass 2 kg and length 1.5 m rotates about an axis passing through its centre, perpendicular to its length. Calculate its moment of inertia.

Solution: Using $I = ML^2/12$, $I = (2 \times 1.5^2) / 12 = (2 \times 2.25) / 12 = 4.5 / 12 = 0.375$ kg·m².

Worked Example 2:
A circular disc of mass 5 kg and radius 0.4 m rotates about an axis through its centre, perpendicular to its plane. Calculate its moment of inertia and its radius of gyration about this axis.

Solution: Using $I = MR^2/2$, $I = (5 \times 0.4^2) / 2 = (5 \times 0.16) / 2 = 0.8 / 2 = 0.4$ kg·m². Using $K = \sqrt{(I/M)}$, $K = \sqrt{(0.4/5)} = \sqrt{0.08} \approx 0.283$ m.

Worked Example 3:
A circular ring of mass 1 kg and radius 0.2 m rotates about an axis through its centre, perpendicular to its plane. Calculate its moment of inertia.

Solution: Using $I = MR^2$, $I = 1 \times 0.2^2 = 1 \times 0.04 = 0.04$ kg·m².

Worked Example 4:
A solid sphere of mass 3 kg and radius 0.1 m rotates about a diameter. Calculate its moment of inertia.

Solution: Using $I = 2MR^2/5$, $I = (2 \times 3 \times 0.1^2) / 5 = (2 \times 3 \times 0.01) / 5 = 0.06 / 5 = 0.012$ kg·m².

6. Multiple Choice Questions (MCQs)

Answer the following questions by selecting the most appropriate option.

1. Moment of inertia of a rigid body about an axis is the rotational analogue of:

A) Force

B) Mass

C) Velocity

D) Momentum

2. Moment of inertia represents a body's ability to resist a change in its:

A) Linear motion

B) Rotational motion

C) Shape

D) Temperature

3. The SI unit of moment of inertia is:

A) kg·m

B) $kg \cdot m^2$

C) kg/m^2

D) N·m

4. The radius of gyration of a rigid body about a given axis is defined by the relation:

A) I = MK

B) $I = MK^2$

C) $I = M/K^2$

D) I = K/M

5. Radius of gyration is best described as:

A) The actual radius of the body

B) The distance from the axis at which the entire mass, if concentrated, would give the same moment of inertia

C) Always equal to the geometrical radius of the body

D) A quantity with units of $kg \cdot m^2$

6. According to the theorem of parallel axes, the moment of inertia about any axis parallel to an axis through the centre of mass is:

A) Always less than the moment of inertia about the centre of mass

B) Equal to the moment of inertia about the centre of mass plus Md^2

C) Equal to the moment of inertia about the centre of mass minus Md^2

D) Independent of the distance between the axes

7. The theorem of perpendicular axes applies to:

A) Any three-dimensional rigid body

B) Only spherical bodies

C) Plane laminar (flat, two-dimensional) bodies only

D) Only rods

8. According to the theorem of perpendicular axes, for a plane lamina, Iz is equal to:

A) Ix − Iy

B) $I_x \times I_y$

C) $I_x + I_y$

D) I_x / I_y

9. The moment of inertia of a uniform rod of mass M and length L about an axis through its centre, perpendicular to its length, is:

A) $ML^2/3$

B) $ML^2/12$

C) $ML^2/2$

D) ML^2

10. The moment of inertia of a solid disc of mass M and radius R about an axis through its centre, perpendicular to its plane, is:

A) MR^2

B) $MR^2/2$

C) $MR^2/4$

D) $2MR^2/5$

11. The moment of inertia of a circular ring of mass M and radius R about an axis through its centre, perpendicular to its plane, is:

A) $MR^2/2$

B) $MR^2/4$

C) MR^2

D) $2MR^2/5$

12. The moment of inertia of a solid sphere of mass M and radius R about a diameter is:

A) MR^2

B) $2MR^2/5$

C) $2MR^2/3$

D) $MR^2/2$

13. The moment of inertia of a hollow (thin) spherical shell of mass M and radius R about a diameter is:

A) $2MR^2/5$

B) $2MR^2/3$

C) MR^2

D) $MR^2/2$

14. For the same mass and radius, which body has the greater moment of inertia about a central axis — a solid sphere or a hollow sphere?

A) Solid sphere

B) Hollow sphere

C) Both are equal

D) Cannot be compared

15. A body has a greater moment of inertia about a given axis when its mass is distributed:

A) Closer to the axis
B) Farther from the axis
C) Uniformly regardless of distance
D) Only along the axis

Answer Key

1. B) Mass | 2. B) Rotational motion | 3. B) kg·m² | 4. B) $I = MK^2$ | 5. B) The distance from the axis at which the entire mass, if concentrated, would give the same moment of inertia | 6. B) Equal to the moment of inertia about the centre of mass plus Md^2 | 7. C) Plane laminar (flat, two-dimensional) bodies only | 8. C) $I_x + I_y$ | 9. B) $ML^2/12$ | 10. B) $MR^2/2$ | 11. C) MR^2 | 12. B) $2MR^2/5$ | 13. B) $2MR^2/3$ | 14. B) Hollow sphere | 15. B) Farther from the axis

7. Broad / Long Answer and Numerical Questions

1. Define moment of inertia of a rigid body about an axis of rotation. Explain its physical significance, drawing an analogy with mass in translational motion.

2. Define radius of gyration of a rigid body. Derive the relation between moment of inertia, mass, and radius of gyration.

3. State the theorem of parallel axes for moment of inertia (statement only, no derivation required).

4. State the theorem of perpendicular axes for moment of inertia, and mention the type of body to which it applies (statement only, no derivation required).

5. Write the formulae for the moment of inertia of a uniform rod about an axis through its centre and about an axis through one end, both perpendicular to its length.

6. Write the formulae for the moment of inertia of a circular disc and a circular ring, each about an axis through the centre, perpendicular to the plane of the body.

7. Write the formulae for the moment of inertia of a solid sphere and a hollow spherical shell, each about a diameter, and state which of the two has the greater moment of inertia for the same mass and radius.

8. Explain why the moment of inertia of a body depends not only on its mass but also on the distribution of that mass relative to the axis of rotation.

9. A uniform rod of mass 2 kg and length 1.5 m rotates about an axis passing through its centre, perpendicular to its length. Calculate its moment of inertia.

10. A circular disc of mass 5 kg and radius 0.4 m rotates about an axis through its centre, perpendicular to its plane. Calculate its moment of inertia and its radius of gyration about this axis.