Applied Physics 1

Rotational Motion, Torque and Angular Momentum

1. Translational and Rotational Motion

1.1 Translational Motion

Translational motion is the motion of a body in which every particle of the body undergoes the same displacement, in the same direction, during any given interval of time. In translational motion, the orientation of the body does not change, and the body may move along a straight line (rectilinear translation) or along a curved path (curvilinear translation), as long as every particle moves in an identical manner.

Examples of translational motion:

- A car moving along a straight road.
- A book sliding across a table.
- A ball falling freely under gravity.
- A train moving along a curved railway track (curvilinear translation).

1.2 Rotational Motion

Rotational motion is the motion of a body in which the body turns about a fixed axis, called the axis of rotation, such that every particle of the body moves in a circular path around this axis. Unlike translational motion, different particles of a rotating body move through different linear distances in the same time, depending on their distance from the axis of rotation, although they all sweep through the same angle.

Examples of rotational motion:

- A ceiling fan rotating about its central axis.
- A spinning top rotating about its own axis.
- The Earth rotating about its polar axis, causing day and night.
- A wheel rotating about its axle.

In many real situations, a body can exhibit both translational and rotational motion simultaneously. For example, a rolling wheel of a moving vehicle both rotates about its own axis and translates forward along the road, combining both types of motion.

The table below compares the two types of motion:

Aspect	Translational Motion	Rotational Motion
Definition	Motion in which every particle of a body moves through the same distance in the same direction	Motion in which a body turns about a fixed axis, with different particles moving through different distances
Path of particles	All particles follow parallel straight or curved paths	All particles move in circles about the axis of rotation
Key quantities	Displacement, velocity, acceleration, mass	Angular displacement, angular velocity, angular acceleration, moment of inertia
Cause of motion	Force	Torque
Examples	A car moving on a straight road; a ball falling freely	A rotating fan; a spinning top; the Earth rotating about its own axis

2. Torque

Torque is the rotational analogue of force. It is defined as the turning effect (or moment) produced by a force about a fixed point or axis of rotation. Just as a force is required to produce or change the linear motion of a body, a torque is required to produce or change the rotational motion of a body about an axis.

If a force F acts on a body at a perpendicular distance r from the axis of rotation, the torque τ produced by the force is given by:

- $\tau = F \times r$

where r is called the moment arm or lever arm, that is, the perpendicular distance from the axis of rotation to the line of action of the force. The SI unit of torque is the newton-metre (N·m).

Torque depends on both the magnitude of the applied force and the perpendicular distance at which it acts from the axis. This is why a larger torque, and hence a greater turning effect, is produced either by applying a greater force or by applying the same force farther from the axis of rotation — which is the underlying principle behind using a long spanner to loosen a tight nut, or a long handle to operate a door.

3. Angular Momentum

Angular momentum is the rotational analogue of linear momentum. It is defined as the moment of linear momentum of a body about the axis of rotation, and it is a measure of the quantity of rotational motion possessed by a body.

For a body with moment of inertia I rotating with angular velocity ω about a fixed axis, the angular momentum L is given by:

- $L = I\omega$

The SI unit of angular momentum is kilogram-metre squared per second ($kg \cdot m^2/s$), which is also equivalent to joule-second (J·s). Angular momentum is a vector quantity, directed along the axis of rotation.

Here, the moment of inertia I plays the same role in rotational motion that mass plays in translational motion — it represents the rotational inertia of a body, that is, its tendency to resist any change in its rotational motion, and depends on both the mass of the body and how that mass is distributed relative to the axis of rotation.

4. Relation Between Torque and Angular Momentum

Just as force is related to the rate of change of linear momentum ($F = dp/dt$) in translational motion, torque is related to the rate of change of angular momentum in rotational motion. This relation is given by:

- $\tau = dL / dt$

This equation states that the torque acting on a body is equal to the rate at which its angular momentum changes with time. If no external torque acts on a system, the rate of change of its angular momentum is zero, meaning the angular momentum of the system remains constant — this is precisely the basis of the law of conservation of angular momentum, discussed in the next section.

The analogy between translational and rotational quantities is summarised below, which is useful for remembering the corresponding formulae in rotational motion:

Translational Quantity	Rotational Analogue
Displacement (s)	Angular displacement (θ)
Velocity (v)	Angular velocity (ω)
Acceleration (a)	Angular acceleration (α)
Mass (m)	Moment of inertia (I)
Force ($F = ma$)	Torque ($\tau = I\alpha$)
Momentum ($p = mv$)	Angular momentum ($L = I\omega$)

5. Conservation of Angular Momentum (Quantitative)

The law of conservation of angular momentum states that, if no external torque acts on a system, the total angular momentum of the system remains constant. This follows directly from the relation $\tau = dL/dt$: if the net external torque τ is zero, then $dL/dt = 0$, which means L does not change with time.

For a rotating body whose moment of inertia changes from I_1 to I_2 (for example, due to a redistribution of mass), with no external torque acting, the law of conservation of angular momentum can be expressed quantitatively as:

- $I_1 \omega_1 = I_2 \omega_2$

where ω_1 and ω_2 are the angular velocities of the body corresponding to the moments of inertia I_1 and I_2 respectively. This relation shows that if the moment of inertia of a rotating body decreases, its angular velocity must increase correspondingly (and vice versa), so that the product $I\omega$ remains constant.

Worked Example:
A rotating body has a moment of inertia of 4 $kg{\cdot}m^2$ and an angular velocity of 5 rad/s. Its moment of inertia is then reduced to 2 $kg{\cdot}m^2$, with no external torque acting on it. Find its new angular velocity.

Solution: Using conservation of angular momentum, $I_1\omega_1 = I_2\omega_2$. So, $4 \times 5 = 2 \times \omega_2$, which gives $20 = 2\omega_2$, so $\omega_2 = 10$ rad/s. The angular velocity of the body doubles as its moment of inertia is halved, keeping the angular momentum ($I\omega = 20$ $kg{\cdot}m^2/s$) unchanged.

5.1 Applications of Conservation of Angular Momentum

The law of conservation of angular momentum has several important applications and observed effects in nature and everyday life:

- Spinning ice skater: When a skater spinning with arms outstretched pulls their arms inward, their moment of inertia decreases (since mass is brought closer to the axis of rotation). Since no significant external torque acts on the skater during this action, their angular momentum remains constant, and their angular velocity increases correspondingly — causing them to spin faster.
- Diver performing a somersault: A diver jumping off a diving board curls their body into a tucked position while in the air, reducing their moment of inertia. With no external torque acting on them during free fall, their angular momentum is conserved, so their angular velocity increases, allowing them to complete more somersaults before straightening out again just before entering the water, which increases their moment of inertia and slows their rotation.

- Planetary motion: A planet moving in an elliptical orbit around the Sun moves faster when it is closer to the Sun (where its effective radius, and hence moment of inertia about the Sun, is smaller) and slower when farther away, since the gravitational force acts along the line joining the planet and the Sun and produces no torque about the Sun, keeping the planet's angular momentum constant throughout its orbit.
- Rotating stars (pulsars): When a massive star collapses under gravity to form a much smaller, denser neutron star, its moment of inertia decreases dramatically. Since angular momentum is conserved during the collapse, the star's rotational speed increases enormously, resulting in the extremely fast spin rates observed in pulsars.
- A person sitting on a rotating stool holding weights: If the person pulls the weights in close to their body while spinning, their moment of inertia decreases and, by conservation of angular momentum, their rate of spin increases; extending the arms outward again slows the spin.

6. Multiple Choice Questions (MCQs)

Answer the following questions by selecting the most appropriate option.

1. In translational motion, all particles of a body:

A) Move in circles about a fixed axis
B) Move through the same distance in the same direction
C) Remain stationary
D) Move with different velocities in random directions

2. Which of the following is an example of pure rotational motion?

A) A car moving on a straight road
B) A ceiling fan rotating about its axis
C) A ball falling freely under gravity
D) A boat moving on a river

3. Torque is defined as the moment of:

A) Mass about an axis
B) Force about an axis or point
C) Velocity about an axis
D) Momentum about an axis

4. The formula for torque is:

A) $\tau = F \times r$
B) $\tau = F / r$
C) $\tau = F + r$
D) $\tau = m \times r$

5. The SI unit of torque is:

A) Joule
B) Newton-metre (N·m)
C) Watt
D) Newton-second

6. Angular momentum is defined as the moment of:

A) Force about an axis
B) Linear momentum about an axis
C) Mass about an axis
D) Velocity about an axis

7. The formula for angular momentum in terms of moment of inertia and angular velocity is:

A) $L = I / \omega$
B) $L = I \times \omega$
C) $L = I + \omega$
D) $L = \omega / I$

8. The relation between torque and angular momentum is:

A) $\tau = dL/dt$
B) $\tau = L \times t$
C) $\tau = L / t^2$
D) $\tau = dL \times dt$

9. The SI unit of angular momentum is:

A) N·m
B) $kg \cdot m^2/s$
C) kg·m/s
D) N·s

10. The law of conservation of angular momentum states that angular momentum remains constant when:

A) No net force acts on the body
B) No net external torque acts on the system
C) The body moves with constant linear velocity
D) The moment of inertia is zero

11. When a spinning ice skater pulls their arms inward, their angular velocity:

A) Decreases
B) Increases
C) Remains the same
D) Becomes zero

12. According to the law of conservation of angular momentum, if the moment of inertia of a rotating body decreases, its angular velocity must:

A) Decrease
B) Increase
C) Remain constant
D) Become zero

13. Which of the following is an application of conservation of angular momentum?

A) A car braking on a straight road
B) A diver curling their body while performing a somersault in the air
C) A ball rolling down an incline
D) A block sliding on a rough surface

14. The planets move faster when nearer to the Sun and slower when farther away, mainly due to:

A) Conservation of linear momentum
B) Conservation of angular momentum
C) Conservation of mechanical energy alone
D) Newton's third law only

15. Torque acting on a body produces:

A) Linear acceleration only
B) Angular acceleration
C) No change in motion
D) A decrease in mass

Answer Key

1. B) Move through the same distance in the same direction | 2. B) A ceiling fan rotating about its axis | 3. B) Force about an axis or point | 4. A) $\tau = F \times r$ | 5. B) Newton-metre (N·m) | 6. B) Linear momentum about an axis | 7. B) $L = I \times \omega$ | 8. A) $\tau = dL/dt$ | 9. B) $kg \cdot m^2/s$ | 10. B) No net external torque acts on the system | 11. B) Increases | 12. B) Increase | 13. B) A diver curling their body while performing a somersault in the air | 14. B) Conservation of angular momentum | 15. B) Angular acceleration

7. Broad / Long Answer and Numerical Questions

1. Distinguish between translational motion and rotational motion, giving two examples of each.

2. Define torque. Derive the relation $\tau = F \times r$, and explain the factors on which the torque produced by a force depends.

3. Define angular momentum. Derive the relation $L = I\omega$ for a rotating body.

4. Establish the relation between torque and angular momentum, $\tau = dL/dt$, and explain its physical significance.

5. State the law of conservation of angular momentum, and explain the condition under which it holds true.

6. Explain, with the help of the law of conservation of angular momentum, why a spinning ice skater spins faster when they pull their arms inward.

7. Explain the application of conservation of angular momentum in the case of a diver performing a somersault in the air.

8. Explain how the law of conservation of angular momentum accounts for the varying orbital speed of a planet as it moves around the Sun in an elliptical orbit.

9. Draw up the analogy between translational and rotational quantities (displacement, velocity, mass, force, momentum) and their rotational counterparts.

10. A rotating body has a moment of inertia of 4 $kg \cdot m^2$ and an angular velocity of 5 rad/s. If its moment of inertia is reduced to 2 $kg \cdot m^2$ with no external torque acting, calculate its new angular velocity using conservation of angular momentum.