CIRCUIT THEORY

Unit 5: Laplace Transform

Diploma / Polytechnic Engineering — Study Notes

5.1 Definition and Properties of Laplace Transform

The Laplace Transform is a mathematical tool that converts a function of time, f(t), into a function of a complex frequency variable 's', denoted as F(s). It is extremely useful in circuit theory because it converts linear differential equations (which describe the behaviour of circuits containing inductors and capacitors) into simple algebraic equations in the 's' domain, which are much easier to solve. Once solved in the s-domain, the inverse Laplace Transform is used to convert the result back into the time domain.

Definition

The Laplace Transform of a function f(t), defined for $t \geq 0$, is given by the integral: $F(s) = L\{f(t)\} = \int_0^\infty f(t)\, e^{-st}\, dt$, where 's' is a complex frequency variable ($s = \sigma + j\omega$), and the integral is evaluated from $t = 0$ to $t = \infty$, provided the integral converges.

- f(t): the given function of time (original function), defined for $t \geq 0$.
- F(s): the Laplace transform of f(t), a function of the complex variable s.
- e^{-st}: the kernel of the transformation, which ensures convergence of the integral for a suitably large real part of s.
- The inverse Laplace Transform, denoted $L^{-1}\{F(s)\} = f(t)$, converts F(s) back to the original time-domain function.

Properties of Laplace Transform

Property	Statement
Linearity	$L\{a \cdot f(t) + b \cdot g(t)\} = a \cdot F(s) + b \cdot G(s)$, for constants a and b — the transform of a sum is the sum of the transforms.
Time scaling	$L\{f(at)\} = (1/a)\, F(s/a)$, for $a > 0$ — scaling the time axis scales the frequency axis inversely.
Frequency (s-domain) shifting	$L\{e^{at} f(t)\} = F(s - a)$ — multiplying a time function by an exponential shifts the s-domain variable.
Time shifting	$L\{f(t - a)\, u(t - a)\} = e^{-as} F(s)$, for $a > 0$ — delaying a function in time multiplies its transform by e^{-as}.
Differentiation (time domain)	$L\{f'(t)\} = sF(s) - f(0)$, and $L\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$ — differentiation in time becomes multiplication by 's' in the s-domain (minus initial condition terms).
Integration (time domain)	$L\{\int_0^t f(\tau)\, d\tau\} = F(s)/s$ — integration in time becomes division by 's' in the s-domain.
Differentiation (s-domain)	$L\{t \cdot f(t)\} = -dF(s)/ds$ — multiplying a time function by 't' corresponds to differentiating its transform with respect to s (with a negative sign).

> *The differentiation property, $L\{f'(t)\} = sF(s) - f(0)$, is the single most important property for circuit analysis, since it directly converts the differential equations of R-L and R-C circuits (which involve di/dt or dv/dt) into simple algebraic equations in 's', while automatically including the initial conditions (initial current or initial capacitor voltage).*

Applications

- Converts differential equations describing electrical circuits into algebraic equations, which are far easier to solve.
- Provides a systematic method to include initial conditions (initial inductor current, initial capacitor voltage) directly in the solution.
- Widely used in control systems, signal processing and communication engineering for system and transfer function analysis.

Limitations

- Laplace Transform methods apply directly to linear, time-invariant systems; they are not directly applicable to non-linear circuit elements.
- The transform exists only for functions that satisfy certain convergence conditions (piecewise continuous, of exponential order); some functions do not have a valid Laplace Transform.

5.2 Laplace Transform of Standard Functions, Initial and Final Value Theorems

Certain standard time-domain functions occur very frequently in circuit analysis — such as the unit step, unit impulse, ramp, exponential, sine and cosine functions. Their Laplace transforms are derived once and then used directly (from standard tables) whenever these functions appear in a circuit problem.

5.2.1 Unit Step Function, u(t)

The unit step function $u(t)$ is defined as $u(t) = 0$ for $t < 0$, and $u(t) = 1$ for $t \geq 0$. It represents the sudden application of a constant value (such as switching on a D.C. source) at $t = 0$.

- Derivation: $L\{u(t)\} = \int_0^\infty 1 \cdot e^{-st}\, dt = [-e^{-st}/s]_0^\infty = 0 - (-1/s) = 1/s$.
- Result: $L\{u(t)\} = 1/s$.

5.2.2 Unit Impulse Function, δ(t)

The unit impulse function (Dirac delta function) $\delta(t)$ is an idealised function that is zero everywhere except at $t = 0$, where it is infinitely large, such that its total area under the curve equals 1. It represents a very short, very large pulse, such as a sudden surge or a sharp switching transient.

- Using the sifting property of the impulse function, $\int_0^\infty \delta(t)\, e^{-st}\, dt = e^{-s \cdot 0} = 1$.
- Result: $L\{\delta(t)\} = 1$.

5.2.3 Ramp Function, r(t) = t

The unit ramp function is defined as $r(t) = t$ for $t \geq 0$ (and 0 for $t < 0$). It represents a quantity that increases linearly with time, such as a linearly rising test voltage.

- Derivation: $L\{t\} = \int_0^\infty t \cdot e^{-st}\, dt$, which on integration by parts gives $1/s^2$.
- Result: $L\{t\} = 1/s^2$.

5.2.4 Exponential Function, e^(−at)

- Derivation: $L\{e^{-at}\} = \int_0^\infty e^{-at}\, e^{-st}\, dt = \int_0^\infty e^{-(s+a)t}\, dt = 1/(s + a)$, for $s > -a$.
- Result: $L\{e^{-at}\} = 1/(s + a)$.

5.2.5 Sine Function, sin(ωt)

- Using the standard integral (or Euler's formula for sin ωt in terms of exponentials): $L\{\sin(\omega t)\} = \omega / (s^2 + \omega^2)$.

5.2.6 Cosine Function, cos(ωt)

- Using the standard integral (or Euler's formula for cos ωt in terms of exponentials): $L\{\cos(\omega t)\} = s / (s^2 + \omega^2)$.

Table of Standard Laplace Transforms

f(t)	F(s) = L{f(t)}
Unit step, u(t)	1/s
Unit impulse, δ(t)	1
Ramp, t	$1/s^2$
Exponential, e^{-at}	1/(s + a)
Sine, sin(ωt)	$\omega/(s^2 + \omega^2)$
Cosine, cos(ωt)	$s/(s^2 + \omega^2)$

5.2.7 Initial Value Theorem

The Initial Value Theorem allows the initial value of a time-domain function, $f(0^+)$, to be found directly from its Laplace transform F(s), without performing the inverse transform, by taking the limit as s tends to infinity.

> *Initial Value Theorem: $f(0^+) = \lim_{t\to 0} f(t) = \lim_{s\to\infty} [s \cdot F(s)]$, provided the limit exists.*

5.2.8 Final Value Theorem

The Final Value Theorem allows the final (steady-state) value of a time-domain function, f(∞), to be found directly from its Laplace transform F(s), without performing the inverse transform, by taking the limit as s tends to zero.

> *Final Value Theorem: $f(\infty) = \lim_{t\to\infty} f(t) = \lim_{s\to 0} [s \cdot F(s)]$, provided all poles of sF(s) lie in the left half of the s-plane (i.e., the system is stable and reaches a steady value).*

Applications

- Standard transform tables allow quick conversion of common circuit excitation and response functions between the time and s-domains without repeated integration.
- Initial and Final Value Theorems allow quick verification of circuit behaviour immediately after switching ($t = 0^+$) and after reaching steady state ($t \to \infty$), directly from the s-domain expression, saving the effort of a full inverse transform.

Limitations

- The Final Value Theorem gives incorrect (meaningless) results if the system is unstable or if the response does not settle to a finite steady value (e.g., for pure oscillatory or growing responses), so the pole location condition must always be checked first.
- Standard transform tables only cover a limited set of common functions; more complex or piecewise functions require additional properties (like time-shifting) or must be broken down into combinations of standard functions.

5.3 Applications of Laplace Transform for Solving Differential Equations of Simple Electrical Circuits

One of the most powerful uses of the Laplace Transform in circuit theory is to solve the differential equations that describe the transient behaviour of R-L and R-C circuits, by transforming them into simple algebraic equations in the s-domain. This avoids the need to solve differential equations directly using classical methods.

5.3.1 Transformed (s-Domain) Circuit Elements

To analyse a circuit directly in the s-domain, each circuit element is represented by its equivalent s-domain impedance, and any initial energy stored in inductors or capacitors is represented by an additional equivalent source:

Element	s-Domain Representation
Resistor (R)	s-domain impedance remains R (unchanged), since a resistor has no time-derivative or integral relationship between voltage and current.
Inductor (L)	s-domain impedance = sL, in series with a voltage source of value L·i(0) (representing the initial inductor current i(0)), or equivalently modelled with a parallel current source i(0)/s.
Capacitor (C)	s-domain impedance = 1/(sC), in series with a voltage source of value vC(0)/s (representing the initial capacitor voltage vC(0)), or equivalently modelled with a parallel current source C·vC(0).

5.3.2 General Procedure for Solving Circuits Using Laplace Transform

1. Write the differential (integro-differential) equation describing the given circuit in the time domain, using Kirchhoff's Voltage/Current Law.
2. Take the Laplace Transform of both sides of the equation, applying the standard transform table and the differentiation/integration properties, and substituting the given initial conditions (initial current or initial capacitor voltage).
3. Rearrange the resulting algebraic equation to solve for the desired quantity (such as I(s), the s-domain current) in terms of s.
4. Simplify I(s) (or V(s)) using partial fraction expansion, if necessary, so that it is expressed as a sum of simple standard terms matching entries in the Laplace transform table.
5. Take the inverse Laplace Transform of each term to obtain the final time-domain solution, i(t) (or v(t)).

5.3.3 Example — R-L Circuit Solved Using Laplace Transform

Consider a series R-L circuit connected to a D.C. voltage source V through a switch closed at t = 0, with zero initial current (i(0) = 0). The time-domain KVL equation is: $V\cdot u(t) = i\cdot R + L(di/dt)$.

1. Taking the Laplace Transform of both sides: $V/s = R\cdot I(s) + L[sI(s) - i(0)]$. Since $i(0) = 0$, this simplifies to $V/s = R\cdot I(s) + L\cdot s\cdot I(s)$.
2. Rearranging: $V/s = I(s)(R + sL)$, so $I(s) = V / [s(R + sL)] = (V/L) / [s(s + R/L)]$.
3. Expressing this using partial fractions: $I(s) = (V/R)\,[1/s - 1/(s + R/L)]$.
4. Taking the inverse Laplace Transform (using $L^{-1}\{1/s\} = 1$ and $L^{-1}\{1/(s+a)\} = e^{-at}$): $i(t) = (V/R)\,[1 - e^{-Rt/L}]$.

> *This result, $i(t) = (V/R)[1 - e^{-Rt/L}]$, is exactly the same growth-of-current equation derived earlier in Chapter 4 using classical differential equation methods — confirming that the Laplace Transform method gives the identical (correct) time-domain result through a purely algebraic s-domain procedure.*

Applications

- Systematic, algebra-based solution of transient response (both natural and forced response) of R-L, R-C and R-L-C circuits with any initial conditions.
- Analysis of circuits subjected to complex excitation waveforms (step, impulse, ramp, sinusoidal) using the same standard procedure.
- Forms the basis of transfer function analysis, used extensively in control systems and filter design.

Limitations

- The method assumes linear, time-invariant circuit elements; it cannot be applied directly to circuits with non-linear components.
- Partial fraction expansion can become tedious for higher-order circuits (with many energy-storage elements) or for repeated/complex poles.

5.4 Related Numerical Problems

This section presents solved numerical examples on finding Laplace transforms of standard functions, applying the initial and final value theorems, and solving a simple circuit differential equation using the Laplace Transform method.

Numerical on Laplace Transform of a Combination of Functions

Example 1: Laplace transform using linearity and standard transform table

Given: Find the Laplace Transform of $f(t) = 5 + 3e^{-2t} + 4\sin(3t)$.

Solution

Step 1 (Apply linearity): $L\{f(t)\} = 5 \cdot L\{1\} + 3 \cdot L\{e^{-2t}\} + 4 \cdot L\{\sin(3t)\}$.

Step 2 (Use standard transforms): $L\{1\} = 1/s$, $L\{e^{-2t}\} = 1/(s+2)$, and $L\{\sin(3t)\} = 3/(s^2+9)$.

Step 3 (Substitute and combine): $F(s) = 5/s + 3/(s+2) + 4\times[3/(s^2+9)] = 5/s + 3/(s+2) + 12/(s^2+9)$.

Answer: $F(s) = 5/s + 3/(s+2) + 12/(s^2+9)$

Numerical on Inverse Laplace Transform Using Partial Fractions

Example 2: Inverse Laplace transform of a rational function

Given: Find the inverse Laplace transform of $F(s) = 10 / [s(s+5)]$.

Solution

Step 1 (Partial fraction expansion): Write $F(s) = 10/[s(s+5)] = A/s + B/(s+5)$. Multiplying both sides by $s(s+5)$: $10 = A(s+5) + Bs$.

Step 2 (Find A): Put $s = 0$: $10 = A(0+5) + 0$, so $A = 2$.

Step 3 (Find B): Put $s = -5$: $10 = 0 + B(-5)$, so $B = -2$.

Step 4 (Write F(s) in expanded form and take inverse transform): $F(s) = 2/s - 2/(s+5)$; using $L^{-1}\{1/s\} = 1$ and $L^{-1}\{1/(s+a)\} = e^{-at}$: $f(t) = 2 - 2e^{-5t}$.

Answer: $f(t) = 2 - 2e^{-5t} = 2(1 - e^{-5t})$

Numerical on Initial and Final Value Theorems

Example 3: Applying the Initial Value and Final Value Theorems

Given: For $F(s) = 20 / [s(s+4)]$, find the initial value $f(0^+)$ and the final value $f(\infty)$ using the Initial Value Theorem and Final Value Theorem, without finding the complete inverse transform.

Solution

Step 1 (Initial value): $f(0^+) = \lim(s\to\infty)\ [s\cdot F(s)] = \lim(s\to\infty)\ [s \times 20/(s(s+4))] = \lim(s\to\infty)\ [20/(s+4)] = 0$ (as $s\to\infty$, the denominator becomes very large).

Step 2 (Check stability for Final Value Theorem): $sF(s) = 20/(s+4)$, which has a pole at $s = -4$ (in the left half of the s-plane), so the Final Value Theorem is valid.

Step 3 (Final value): $f(\infty) = \lim(s\to 0)\ [s\cdot F(s)] = \lim(s\to 0)\ [20/(s+4)] = 20/4 = 5$.

> **Answer: $f(0^+) = 0$, and $f(\infty) = 5$**

Numerical on R-C Circuit Using Laplace Transform

Example 4: Solving an R-C charging circuit using the Laplace Transform method

Given: A series R-C circuit ($R = 100\ \Omega$, $C = 10\ \mu F$) is connected to a D.C. supply of 50 V through a switch closed at $t = 0$, with the capacitor initially uncharged ($vC(0) = 0$). Using the Laplace Transform method, find the expression for the charging current $i(t)$.

Solution

Step 1 (Time-domain KVL equation): $50\cdot u(t) = i\cdot R + vC$, where $vC = (1/C)\int i\ dt$.

Step 2 (Take Laplace Transform): $50/s = I(s)\cdot R + I(s)/(sC)$ (since $vC(0) = 0$, there is no extra initial-condition source term).

Step 3 (Rearrange): $50/s = I(s)\ [R + 1/(sC)] = I(s) \times [(sRC + 1)/(sC)]$, so $I(s) = 50C / (sRC + 1) = (50/R) / [s + 1/(RC)]$.

Step 4 (Substitute values): $1/(RC) = 1/(100 \times 10\times10^{-6}) = 1/(10^{-3}) = 1000$, and $50/R = 50/100 = 0.5$. So $I(s) = 0.5 / (s + 1000)$.

Step 5 (Take inverse Laplace Transform): Using $L^{-1}\{1/(s+a)\} = e^{-at}$: $i(t) = 0.5 \times e^{-1000t}$ A.

> **Answer: $i(t) = 0.5\ e^{-1000t}$ A (which matches the classical charging-current formula $i(t) = (V/R)\ e^{-t/RC}$, since $V/R = 50/100 = 0.5$ A and $1/RC = 1000$)**

Multiple Choice Questions (MCQs)

Choose the correct option for each question. The correct answer is shown in bold within the options, and a separate answer key is also given at the end.

Q1. The Laplace Transform of a function f(t) is defined as:

a) $\int_0^\infty f(t)\ e^{st}\ dt$
b) $\int_0^\infty f(t)\ e^{-st}\ dt$
c) $\int_{-\infty}^\infty f(t)\ dt$
d) $\int_0^\infty f(t)\ dt$

Q2. The Laplace Transform of the unit step function u(t) is:

a) 1
b) 1/s
c) $1/s^2$
d) s

Q3. The Laplace Transform of the unit impulse function δ(t) is:

a) 0
b) 1/s
c) 1
d) s

Q4. The Laplace Transform of e^(−at) is:

a) 1/(s−a)
b) 1/(s+a)
c) a/(s+a)
d) s/(s+a)

Q5. The Laplace Transform of sin(ωt) is:

a) $s/(s^2+\omega^2)$
b) $\omega/(s^2+\omega^2)$
c) $1/(s+\omega)$
d) ω/s^2

Q6. The Laplace Transform of cos(ωt) is:

a) $\omega/(s^2+\omega^2)$
b) $s/(s^2+\omega^2)$
c) $1/(s^2+\omega^2)$
d) s/ω

Q7. According to the differentiation property, L{f'(t)} equals:

a) F(s)/s
b) sF(s) + f(0)
c) sF(s) − f(0)
d) $s^2F(s)$

Q8. The Initial Value Theorem states that $f(0^+)$ equals:

a) lim(s→0) sF(s)
b) lim(s→∞) sF(s)
c) lim(s→∞) F(s)/s
d) lim(s→0) F(s)

Q9. The Final Value Theorem states that f(∞) equals:

a) lim(s→∞) sF(s)
b) lim(s→0) sF(s)
c) lim(s→0) F(s)
d) lim(s→∞) F(s)

Q10. In s-domain circuit analysis, the impedance of an inductor L is represented as:

a) R
b) 1/(sC)
c) sL
d) 1/(sL)

Answer Key

1-b | 2-b | 3-c | 4-b | 5-b | 6-b | 7-c | 8-b | 9-b | 10-c

Broad / Long Answer Questions

1. Define the Laplace Transform. Explain the linearity, time-shifting and differentiation properties with their mathematical statements.

2. Derive the Laplace Transform of the unit step function and the unit impulse function.
3. Derive the Laplace Transform of the exponential function e^(−at) and the ramp function t.
4. Derive the Laplace Transform of sin(ωt) and cos(ωt).
5. State and explain the Initial Value Theorem and Final Value Theorem, along with the condition required for the validity of the Final Value Theorem.
6. Explain how a resistor, an inductor and a capacitor are represented in the s-domain for Laplace-based circuit analysis, including the treatment of initial conditions.
7. Explain the step-by-step procedure for solving the differential equation of an electrical circuit using the Laplace Transform method.
8. A series R-L circuit with zero initial current is connected to a D.C. voltage source V at t = 0. Using the Laplace Transform method, derive the expression for the current i(t).
9. A series R-C circuit with an initially uncharged capacitor is connected to a D.C. voltage source V at t = 0. Using the Laplace Transform method, derive the expression for the charging current i(t).
10. Explain the advantages of using the Laplace Transform method over the classical differential equation method for solving transient circuit problems.

— End of Notes —