Discrete Mathematics

Chapter 6: Counting Techniques

Engineering Notes for Semester Examination

6.1 Principle of Inclusion and Exclusion

In many counting problems, the sets whose elements we wish to count are not disjoint — they overlap with one another. If we simply add the sizes of such overlapping sets, elements that lie in more than one set get counted more than once. The Principle of Inclusion and Exclusion (PIE) is a systematic technique that corrects this over-counting by alternately adding and subtracting the sizes of the sets and their intersections, so that every element is counted exactly once.

Statement of the Principle

For two finite sets A and B:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Here $|A \cap B|$ is subtracted once because the elements common to A and B were counted twice when $|A|$ and $|B|$ were added separately.

For three finite sets A, B and C:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

The pairwise intersections are subtracted because their elements were counted twice, and the triple intersection $|A \cap B \cap C|$ is added back because it was subtracted three times (once in each pairwise term) after being added three times (once in each single set), so it must be restored once.

In general, for n finite sets $A_1, A_2, \ldots, A_n$, the principle states:

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = \Sigma|A_i| - \Sigma|A_i \cap A_j| + \Sigma|A_i \cap A_j \cap A_k| - \ldots + (-1)^{n+1}|A_1 \cap A_2 \cap \ldots \cap A_n|$$

That is, we alternately add the sums of the sizes of single sets, subtract the sums of the sizes of pairwise intersections, add the sums of the sizes of triple intersections, and so on, with the sign alternating as the number of sets in the intersection increases.

Complementary form: If U is a universal set with $|U| = n$ and $A_1, A_2, \ldots, A_k$ are subsets of U, the number of elements in U that lie in *none* of the sets is:

$$|A_1' \cap A_2' \cap \ldots \cap A_k'| = |U| - |A_1 \cup A_2 \cup \ldots \cup A_k|$$

This complementary form is extremely useful when it is easier to count elements satisfying none of several properties rather than at least one of them.

Set Theoretic Problems Relating to the Principle

The principle is best understood through worked examples. A few standard categories of problems are discussed below.

Problem 1 — Numbers divisible by given integers

Find the number of integers from 1 to 100 that are divisible by 3 or 5.

Solution: Let A be the set of integers divisible by 3 and B the set of integers divisible by 5, within 1 to 100.

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- $|A| = \lfloor 100/3 \rfloor = 33$
- $|B| = \lfloor 100/5 \rfloor = 20$
- $|A \cap B|$ = numbers divisible by 15 = $\lfloor 100/15 \rfloor = 6$

By the principle of inclusion and exclusion:

$$|A \cup B| = 33 + 20 - 6 = 47$$

Hence 47 integers between 1 and 100 are divisible by 3 or 5.

Problem 2 — Students opting for subjects

In a class of 100 students, 42 take Mathematics, 38 take Physics, 30 take Chemistry, 12 take Mathematics and Physics, 10 take Mathematics and Chemistry, 8 take Physics and Chemistry, and 5 take all three subjects. Find the number of students taking at least one subject, and the number taking none.

Solution: Using the three-set formula,

$$|M \cup P \cup C| = 42+38+30 - 12-10-8 + 5 = 85$$

So 85 students take at least one of the three subjects, and the remaining 100 − 85 = 15 students take none of them.

Problem 3 — Onto (surjective) functions and derangements

Inclusion-exclusion is also the standard tool for two important counting results that frequently appear in examinations:

- **Number of onto functions** from a set of m elements to a set of n elements ($m \geq n$) is obtained by taking all m^n functions and excluding, using PIE, those that miss at least one element of the codomain:

$$\text{Number of onto functions} = \sum_{k=0}^{n} (-1)^k\, C(n,k)\, (n-k)^m$$

- **Number of derangements** of n objects (permutations in which no object appears in its original position) is derived by excluding, via PIE, all permutations that fix at least one point:

$$D_n = n!\,[1 - 1/1! + 1/2! - 1/3! + \ldots + (-1)^n/n!]$$

Both results follow the same alternating add–subtract pattern that defines the principle, and are useful in problems on distributing distinct objects into distinct boxes with restrictions.

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6.2 Mathematical Induction

Concept of Induction

Mathematical induction is a method of proof used to establish that a statement P(n) is true for every natural number n (usually $n \geq 1$ or $n \geq 0$). It rests on a simple and intuitive idea, often compared to a row of falling dominoes: if the first domino falls, and each falling domino is guaranteed to knock down the next one, then every domino in the row will eventually fall. In the same way, if a statement is true for the smallest case and its truth for any case guarantees its truth for the next case, then the statement is true for all cases from the smallest one onward.

Induction is not a method of discovering a formula — it is a method of proving that a formula or statement, once conjectured, holds for all values of n in its domain.

Statement of the Principle of Mathematical Induction

Let P(n) be a statement involving a natural number n. P(n) is true for all $n \geq n_0$ if the following two conditions hold:

1. **Basis step (or base case):** $P(n_0)$ is true, i.e. the statement holds for the smallest value of n under consideration (commonly $n_0 = 0$ or $n_0 = 1$).
2. **Inductive step:** For every $k \geq n_0$, if P(k) is true (this assumption is called the *induction hypothesis*), then P(k+1) is also true.

If both steps are established, the principle of mathematical induction allows us to conclude that P(n) is true for every $n \geq n_0$.

Principle of Strong (Complete) Induction

A useful variant is the principle of strong induction, where the inductive step assumes that $P(n_0)$, $P(n_0+1)$, …, P(k) are all true, and uses this stronger assumption to prove P(k+1). Strong induction is logically equivalent to ordinary (weak) induction but is often more convenient when the proof of P(k+1) depends on more than just the immediately preceding case, such as in problems involving recurrence relations.

Application of the Principle of Induction in Various Problems

Mathematical induction is used across discrete mathematics to prove summation formulas, divisibility results, inequalities, and properties of recursively defined sequences. A few representative problems are given below.

Problem 1 — Sum of the first n natural numbers

Prove that $1 + 2 + 3 + \ldots + n = n(n+1)/2$ for all $n \geq 1$.

Basis step: For n = 1, LHS = 1 and RHS = 1(2)/2 = 1. The statement holds.

Inductive step: Assume P(k) is true, i.e. $1+2+\ldots+k = k(k+1)/2$. Adding (k+1) to both sides:

$$1+2+\ldots+k+(k+1) = k(k+1)/2 + (k+1) = (k+1)(k+2)/2$$

which is exactly P(k+1). Hence, by induction, P(n) is true for all $n \geq 1$.

Problem 2 — Sum of squares

Prove that $1^2 + 2^2 + \ldots + n^2 = n(n+1)(2n+1)/6$ for all $n \geq 1$. The proof follows the same two-step pattern: the base case n = 1 is verified directly, and the inductive step adds $(k+1)^2$ to the assumed formula for k and simplifies algebraically to match the formula for (k+1).

Problem 3 — Divisibility results

Prove that $3^n - 1$ is divisible by 2 for all $n \geq 1$.

Basis step: For $n = 1$, $3^1 - 1 = 2$, which is divisible by 2.

Inductive step: Assume $3^k - 1 = 2m$ for some integer m. Then:

$$3^{k+1} - 1 = 3 \cdot 3^k - 1 = 3(3^k - 1) + 2 = 3(2m) + 2 = 2(3m+1)$$

which is divisible by 2. Hence the result holds for all $n \geq 1$ by induction.

Problem 4 — Inequalities

Prove that $2^n > n$ for all $n \geq 1$. The base case $n = 1$ gives $2 > 1$, which is true. Assuming $2^k > k$, we get $2^{k+1} = 2 \cdot 2^k > 2k \geq k+1$ (since $k \geq 1$), establishing $P(k+1)$ and completing the proof.

Such inequality and divisibility proofs, along with formulas for sums of series, are the most common examination applications of the principle of mathematical induction.

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6.3 Recurrence Relation

Definition

A recurrence relation for a sequence $a_0, a_1, a_2, \ldots$ is an equation that expresses a term a_n of the sequence in terms of one or more of its preceding terms ($a_{n-1}, a_{n-2}, \ldots$), together with a set of initial conditions that specify the values of the first few terms. Once the recurrence and the initial conditions are known, every term of the sequence can, in principle, be computed step by step.

Order: The order of a recurrence relation is the difference between the largest and smallest subscripts appearing in it. A relation expressing a_n in terms of a_{n-1} and a_{n-2} is of order 2 (second order).

Examples

Fibonacci sequence: The Fibonacci sequence is defined by the second-order recurrence relation

$$F_n = F_{n-1} + F_{n-2}, \ \ n \geq 2, \ \ \text{with } F_0 = 0, F_1 = 1$$

which generates the sequence 0, 1, 1, 2, 3, 5, 8, 13, 21, … and models growth patterns seen in nature and in the analysis of algorithms.

Other examples: The factorial function satisfies the first-order recurrence $n! = n\cdot(n-1)!$ with $0! = 1$. The Tower of Hanoi puzzle with n disks satisfies $H_n = 2H_{n-1} + 1$ with $H_1 = 1$, giving the number of moves required to transfer n disks. Compound interest, $a_n = a_{n-1}(1+r)$, is another familiar first-order recurrence.

Linear Recurrence Relations with Constant Coefficients

A linear recurrence relation with constant coefficients of order k has the general form:

$$a_n + c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k} = f(n)$$

where $c_1, c_2, \ldots, c_k$ are constants (with $c_k \neq 0$) and f(n) is a known function of n. If f(n) = 0 for all n, the relation is called homogeneous; otherwise it is non-homogeneous, and f(n) is referred to as the forcing term or the non-homogeneous part.

Homogeneous Solutions

To solve the homogeneous relation $a_n + c_1a_{n-1} + \ldots + c_ka_{n-k} = 0$, we assume a trial solution of the form $a_n = r^n$ and substitute it into the relation. Dividing through by r^{n-k} yields the characteristic equation:

$$r^k + c_1r^{k-1} + c_2r^{k-2} + \ldots + c_k = 0$$

The form of the general (homogeneous) solution depends on the nature of the roots $r_1, r_2, \ldots, r_k$ of this equation:

- **Distinct real roots:** If all roots are real and distinct, the general solution is $a_n^{(h)} = A_1r_1^n + A_2r_2^n + \ldots + A_kr_k^n$, where $A_1, \ldots, A_k$ are arbitrary constants determined later from the initial conditions.
- **Repeated real roots:** If a root r occurs with multiplicity m, its contribution to the general solution is $(A_1 + A_2n + A_3n^2 + \ldots + A_mn^{m-1})\, r^n$.
- **Complex roots:** Complex roots occur in conjugate pairs $\alpha \pm i\beta$ and contribute terms of the form $\rho^n(A \cos n\theta + B \sin n\theta)$, where $\rho = \sqrt{(\alpha^2+\beta^2)}$ and $\theta = \tan^{-1}(\beta/\alpha)$.

Particular Solutions

A particular solution $a_n^{(p)}$ is any one solution of the complete (non-homogeneous) relation that satisfies the equation for the given f(n). It is usually found by the method of undetermined coefficients, choosing a trial form based on the nature of f(n):

- If f(n) is a constant, try a constant $a_n^{(p)} = A$.
- If f(n) is a polynomial of degree d, try a polynomial of degree d with undetermined coefficients.
- If $f(n) = \beta \cdot d^n$ (an exponential), try $a_n^{(p)} = A \cdot d^n$, provided d is not already a root of the characteristic equation.
- If d happens to be a root of the characteristic equation (of multiplicity m), the trial solution is multiplied by n^m, i.e. $a_n^{(p)} = A \cdot n^m \cdot d^n$, to avoid duplication with the homogeneous solution.

The trial solution is substituted back into the original recurrence relation, and the undetermined coefficients are found by comparing corresponding terms on both sides.

Total Solution

The total (general) solution of a linear recurrence relation with constant coefficients is the sum of the homogeneous solution and a particular solution:

$$a_n = a_n^{(h)} + a_n^{(p)}$$

The arbitrary constants $A_1, A_2, \ldots, A_k$ that appear in the homogeneous part are not evaluated until the particular solution has been added; they are determined last, by substituting the given initial conditions into the total solution and solving the resulting system of equations.

Problems

Problem 1 — Homogeneous relation with distinct roots

Solve $a_n - 5a_{n-1} + 6a_{n-2} = 0$, given $a_0 = 1$, $a_1 = 3$.

Solution: The characteristic equation is $r^2 - 5r + 6 = 0$, i.e. $(r-2)(r-3) = 0$, giving roots $r_1 = 2$, $r_2 = 3$. Hence:

$$a_n = A(2^n) + B(3^n)$$

Using $a_0 = 1$: $A + B = 1$. Using $a_1 = 3$: $2A + 3B = 3$. Solving these simultaneously gives $A = 0$, $B = 1$, so the solution is $a_n = 3^n$.

Problem 2 — Homogeneous relation with repeated roots

Solve $a_n - 4a_{n-1} + 4a_{n-2} = 0$, given $a_0 = 1$, $a_1 = 4$. The characteristic equation $r^2 - 4r + 4 = 0$ factors as $(r-2)^2 = 0$, giving a repeated root $r = 2$. The general solution is $a_n = (A + Bn)2^n$. Applying the initial conditions gives $A = 1$ and $B = 1$, so $a_n = (1+n)2^n$.

Problem 3 — Non-homogeneous relation

Solve $a_n - 3a_{n-1} = 2$, given $a_0 = 1$.

Homogeneous part: The characteristic equation $r - 3 = 0$ gives $r = 3$, so $a_n^{(h)} = A(3^n)$.

Particular part: Since $f(n) = 2$ is a constant and $r = 1$ is not a root of the characteristic equation, try $a_n^{(p)} = C$. Substituting: $C - 3C = 2 \Rightarrow C = -1$.

Total solution: $a_n = A(3^n) - 1$. Using $a_0 = 1$: $A - 1 = 1 \Rightarrow A = 2$. Hence $a_n = 2(3^n) - 1$.

Problem 4 — Fibonacci-type application

Solve the Fibonacci recurrence $F_n = F_{n-1} + F_{n-2}$ with $F_0 = 0$, $F_1 = 1$. The characteristic equation $r^2 - r - 1 = 0$ gives the irrational roots $r = (1 \pm \sqrt{5})/2$, so the general solution is $F_n = A[(1+\sqrt{5})/2]^n + B[(1-\sqrt{5})/2]^n$. Applying the initial conditions and simplifying gives Binet's formula:

$$F_n = [\ ((1+\sqrt{5})/2)^n - ((1-\sqrt{5})/2)^n\ ] / \sqrt{5}$$

This closed-form expression for the Fibonacci sequence is a classic illustration of how solving a linear recurrence relation with constant coefficients converts a recursive definition into a direct (non-recursive) formula.

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Multiple Choice Questions

Choose the correct option for each of the following questions.

1. For two finite sets A and B, $|A \cup B|$ is equal to:

(a) $|A| + |B|$
(b) $|A| + |B| - |A \cap B|$
(c) $|A| + |B| + |A \cap B|$
(d) $|A| - |B|$

2. In the three-set inclusion-exclusion formula, the term $|A \cap B \cap C|$ is:

(a) Subtracted once
(b) Added once
(c) Added twice
(d) Ignored

3. How many integers from 1 to 50 are divisible by 2 or 5?

(a) 25
(b) 10
(c) 30
(d) 35

4. The number of elements in a universal set U that satisfy none of the given properties is found using:

(a) The union formula directly
(b) The complementary form of inclusion-exclusion
(c) The permutation formula
(d) The pigeonhole principle

5. The formula for the number of derangements D_n of n objects is obtained using:

(a) Mathematical induction
(b) The principle of inclusion and exclusion
(c) The recurrence relation method only
(d) The binomial theorem

6. In the principle of mathematical induction, the first step of the proof is called the:

(a) Inductive hypothesis
(b) Basis step
(c) Recursive step
(d) Closure step

7. Strong induction differs from ordinary (weak) induction in that:

(a) It does not require a base case
(b) It assumes the statement is true for all values up to k, not just k
(c) It only applies to inequalities
(d) It cannot be used to prove divisibility results

8. To prove $1+2+\ldots+n = n(n+1)/2$ by induction, the inductive step adds which term to both sides?

(a) n
(b) k
(c) (k+1)
(d) n^2

9. Mathematical induction is primarily used to:

(a) Discover a new formula

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(b) Prove a statement is true for all natural numbers from a starting point onward
(c) Count the elements of a finite set
(d) Solve recurrence relations directly

10. A recurrence relation expresses a term of a sequence in terms of:
(a) Its position number only
(b) One or more preceding terms
(c) The sum of all terms
(d) A random variable

11. The order of the recurrence relation $a_n = 3a_{n-1} - 2a_{n-3}$ is:
(a) 1
(b) 2
(c) 3
(d) 4

12. The Fibonacci sequence is defined by which recurrence relation?
(a) $F_n = F_{n-1} \times F_{n-2}$
(b) $F_n = F_{n-1} + F_{n-2}$
(c) $F_n = F_{n-1} - F_{n-2}$
(d) $F_n = 2F_{n-1}$

13. A linear recurrence relation with constant coefficients is called homogeneous when:
(a) f(n) is a polynomial
(b) f(n) = 0 for all n
(c) f(n) is exponential
(d) The order is 1

14. If the characteristic equation of a second order homogeneous recurrence relation has a repeated root r, the general solution takes the form:
(a) $A \cdot r^n$
(b) $A \cdot r^n + B \cdot (-r)^n$
(c) $(A + Bn) \cdot r^n$
(d) $A \cdot r^n + B \cdot n^2$

15. The total solution of a linear recurrence relation with constant coefficients is:
(a) The homogeneous solution alone
(b) The particular solution alone
(c) The product of the homogeneous and particular solutions
(d) The sum of the homogeneous solution and a particular solution

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Answer Key

1-(b) 2-(b) 3-(c) 4-(b) 5-(b) 6-(b) 7-(b) 8-(c) 9-(b) 10-(b)

11-(c) 12-(b) 13-(b) 14-(c) 15-(d)

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Broad Questions

1. State the principle of inclusion and exclusion for two and three sets, and explain, with reasoning, why the intersection terms are alternately subtracted and added.

2. In a survey of 150 people, 80 like tea, 70 like coffee, 45 like both. Using the principle of inclusion and exclusion, find how many people like at least one of the two beverages and how many like neither.

3. Using the principle of inclusion and exclusion, derive the formula for the number of derangements of n objects, and explain its significance.

4. Explain the concept of mathematical induction with a suitable real-life analogy. State the basis step and the inductive step of the principle precisely.

5. Using the principle of mathematical induction, prove that $1^2 + 2^2 + 3^2 + \ldots + n^2 = n(n+1)(2n+1)/6$ for all $n \geq 1$.

6. Prove, using mathematical induction, that $2^n > n$ for all $n \geq 1$, and explain how the inductive hypothesis is used in the proof.

7. Distinguish between weak (ordinary) induction and strong (complete) induction, and explain a situation where strong induction is more suitable.

8. Define a recurrence relation with an example, and explain the difference between a homogeneous and a non-homogeneous linear recurrence relation with constant coefficients.

9. Solve the recurrence relation $a_n - 5a_{n-1} + 6a_{n-2} = 0$ with $a_0 = 1$ and $a_1 = 3$, showing the characteristic equation, the homogeneous solution, and the evaluation of the constants.

10. Solve the non-homogeneous recurrence relation $a_n - 3a_{n-1} = 2$ with $a_0 = 1$ by finding the homogeneous solution, a particular solution and the total solution.

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