DISCRETE MATHEMATICS

Chapter 4: Functions

Engineering Notes

4.1 Functions

A function is one of the most fundamental concepts in mathematics and computer science. It describes a special kind of relationship between two sets in which every input is associated with exactly one output. Functions are used to model everything from simple formulas to algorithms and data structures.

Definition of Function

Let A and B be two non-empty sets. A function (or mapping) f from A to B, written $f : A \to B$, is a rule (or relation) that assigns to every element $a \in A$ exactly one element $b \in B$. This element b is called the image of a under f, written $b = f(a)$.

- A function is therefore a special type of relation: it is a subset $f \subseteq A \times B$ such that for every $a \in A$, there is exactly one ordered pair $(a, b) \in f$.
- Two conditions must hold for a relation $f \subseteq A \times B$ to be a function: (i) every element of A must appear as the first element of some pair in f (every element of A has an image), and (ii) no element of A may appear as the first element of two different pairs (the image must be unique).
- If a relation assigns two or more different outputs to the same input, or leaves some input without any output, it is not a function.

Domain, Co-domain and Range of a Function

For a function $f : A \to B$, three sets are of particular importance:

- Domain: the set A of all inputs for which f is defined; every element of the domain must have an image. Denoted Dom(f) = A.
- Co-domain: the set B in which the outputs of f are permitted to lie. The co-domain is fixed as part of the definition of f, whether or not every element of B is actually used as an output.
- Range: the set of all actual images of elements of A under f, i.e., $\text{Range}(f) = \{f(a) \mid a \in A\}$. The range is always a subset of the co-domain: $\text{Range}(f) \subseteq B$, but the two need not be equal.

<u>Examples</u>

Example 1: Let A = {1, 2, 3} and B = {a, b, c, d}, and let f = {(1,a), (2,b), (3,b)}. This is a function since every element of A has exactly one image. Here the domain is A = {1,2,3}, the co-domain is B = {a,b,c,d}, and the range is {a, b} (since d is never used as an image).

Example 2: Let $f : R \to R$ be defined by $f(x) = x^2$. Here the domain is R (all real numbers) and the co-domain is R, but the range is only the set of non-negative real numbers $[0, \infty)$, since x^2 can never be negative.

Example 3: Consider g = {(1,x), (1,y), (2,z)} from A = {1,2,3} to B = {x,y,z}. This is not a function, because the element 1 has two images (x and y), and the element 3 has no image at all.

4.2 Injective, Surjective and Bijective Functions

Functions can be classified according to how their inputs relate to their outputs. The three most important classifications are injective (one-one), surjective (onto), and bijective (one-one and onto) functions.

Injective (One-One) Function

A function $f : A \to B$ is said to be injective (or one-one) if distinct elements of A always have distinct images in B, i.e., for all $a1, a2 \in A$, $a1 \neq a2 \implies f(a1) \neq f(a2)$. Equivalently (the contrapositive form, often easier to use in proofs): $f(a1) = f(a2) \implies a1 = a2$.

- Example: $f : R \to R$ defined by $f(x) = 2x + 1$ is injective, since $f(a1) = f(a2) \implies 2a1+1 = 2a2+1 \implies a1 = a2$.
- Example: $f : R \to R$ defined by $f(x) = x^2$ is not injective, since $f(2) = f(-2) = 4$ but $2 \neq -2$, so two different inputs share the same image.
- If A and B are finite sets and $f : A \to B$ is injective, then $n(A) \leq n(B)$.

Surjective (Onto) Function

A function $f : A \to B$ is said to be surjective (or onto) if every element of the co-domain B is the image of at least one element of A, i.e., for every $b \in B$, there exists some $a \in A$ such that $f(a) = b$. In other words, a function is onto if and only if its range equals its co-domain: Range(f) = B.

- Example: $f : R \to R$ defined by $f(x) = x + 3$ is surjective, since for any $y \in R$, we can find $x = y - 3 \in R$ such that $f(x) = y$.
- Example: $f : R \to R$ defined by $f(x) = x^2$ is not surjective, since no negative number is ever an image (the range $[0,\infty)$ is a proper subset of the co-domain R).
- If A and B are finite sets and $f : A \to B$ is surjective, then $n(A) \geq n(B)$.

Bijective (One-One and Onto) Function

A function $f : A \to B$ is said to be bijective if it is both injective and surjective, i.e., every element of B has exactly one pre-image in A. A bijective function is also called a one-to-one correspondence between A and B, and it always has an inverse function $f^{-1} : B \to A$.

- Example: $f : R \to R$ defined by $f(x) = 2x - 5$ is bijective: it is injective (distinct inputs give distinct outputs) and surjective (every real number y is the image of $x = (y+5)/2$).
- If A and B are finite sets and $f : A \to B$ is bijective, then $n(A) = n(B)$.

<u>Related Problems</u>

Problem 1: Let $f : Z \to Z$ be defined by $f(x) = 3x + 2$. Show that f is injective. Solution: Suppose $f(x1) = f(x2)$. Then $3x1 + 2 = 3x2 + 2$, which gives $3x1 = 3x2$, so $x1 = x2$. Hence f is injective.

Problem 2: Let $f : R \to R$ be defined by $f(x) = 3x + 2$. Show that f is surjective. Solution: Let $y \in R$ be any element of the co-domain. We need $x \in R$ such that $f(x) = y$, i.e., $3x + 2 = y$, so $x = (y-2)/3$, which is a real number. Hence for every $y \in R$ there exists a pre-image, so f is surjective. (Together with Problem 1, this shows f is bijective.)

Problem 3: Let A = {1, 2, 3} and B = {p, q}. Show that no injective function $f : A \to B$ can exist. Solution: Since $n(A) = 3 > n(B) = 2$, by the pigeonhole principle at least two elements of A must map to the same element of B under any function $f : A \to B$. Hence no injective function from A to B is possible.

Problem 4: Let A = {1, 2} and B = {p, q, r}. Show that no surjective function f : A → B can exist. Solution: Since n(A) = 2 < n(B) = 3, the range of any function f : A → B can contain at most 2 elements, so it can never equal the entire co-domain B (which has 3 elements). Hence no surjective function from A to B is possible.

Problem 5: Let f : {1,2,3} → {1,2,3} be defined by f(1) = 2, f(2) = 3, f(3) = 1. Determine whether f is bijective. Solution: All three images (2, 3, 1) are distinct, so f is injective; and since the range {1,2,3} equals the co-domain {1,2,3}, f is also surjective. Hence f is bijective.

Multiple Choice Questions (MCQs)

1. A relation f from A to B is a function if:

a) Some elements of A have no image
b) Every element of A has exactly one image in B
c) Every element of B has an image in A
d) f is symmetric

2. For a function f : A → B, the set A is called the:

a) Range
b) Co-domain
c) Domain
d) Image set

3. The range of a function f : A → B is:

a) Always equal to B
b) The set of actual images of elements of A
c) Always equal to A
d) A subset of A

4. For f : R → R defined by $f(x) = x^2$, the range is:

a) R
b) All negative reals
c) $[0, \infty)$
d) $(-\infty, 0]$

5. A relation g = {(1,x),(1,y),(2,z)} from A={1,2,3} to B={x,y,z} fails to be a function because:

a) 3 has two images
b) 1 has two images and 3 has no image
c) It has too many elements
d) x and y are equal

6. A function f : A → B is injective if:

a) $f(a1) = f(a2) \implies a1 = a2$
b) Every element of B has a pre-image
c) A and B have the same cardinality
d) f is not defined for some element

7. Which function is NOT injective?

a) $f(x) = 2x + 1$
b) $f(x) = x^2$ on R
c) $f(x) = 3x - 5$
d) $f(x) = x/2$

8. A function f : A → B is surjective if:

a) Range(f) = A
b) Range(f) = B
c) f is one-one
d) f has no image

9. If A and B are finite sets and f : A → B is injective, then:

a) $n(A) \geq n(B)$
b) $n(A) \leq n(B)$

c) n(A) = n(B) always
d) n(B) = 0

10. If A and B are finite sets and f : A → B is surjective, then:

a) n(A) ≤ n(B)
b) n(A) ≥ n(B)
c) n(A) = 0
d) f cannot exist

11. A bijective function is one that is:

a) Only injective
b) Only surjective
c) Both injective and surjective
d) Neither injective nor surjective

12. A bijective function always has:

a) No inverse
b) A unique inverse function
c) Two inverse functions
d) An empty range

13. Let A = {1,2,3} and B = {p,q}. An injective function from A to B:

a) Always exists
b) Cannot exist, since n(A) > n(B)
c) Exists only if A = B
d) Is always bijective

14. Let A = {1,2} and B = {p,q,r}. A surjective function from A to B:

a) Always exists
b) Cannot exist, since n(A) < n(B)
c) Is always injective
d) Is always bijective

15. For f : {1,2,3} → {1,2,3} defined by f(1)=2, f(2)=3, f(3)=1, the function is:

a) Injective only
b) Surjective only
c) Neither injective nor surjective
d) Bijective

Answer Key

1-b, 2-c, 3-b, 4-c, 5-b, 6-a, 7-b, 8-b, 9-b, 10-b, 11-c, 12-b, 13-b, 14-b, 15-d

Broad / Long Answer Questions

1. Define a function from set A to set B. Explain, with an example, the two conditions a relation must satisfy in order to be a function.

2. Define the domain, co-domain, and range of a function. If $f : R \to R$ is defined by $f(x) = x^2$, identify its domain, co-domain, and range.

3. Give an example of a relation that is not a function, and explain clearly why it fails to be a function.

4. Define an injective (one-one) function. Prove that $f : R \to R$ defined by $f(x) = 3x - 5$ is injective.

5. Define a surjective (onto) function. Prove that $f : R \to R$ defined by $f(x) = x + 3$ is surjective, and show that $f(x) = x^2$ is not surjective.

6. Define a bijective function. Explain why a bijective function always possesses an inverse function.

7. Let $A = \{1,2,3\}$ and $B = \{p,q\}$. Prove, using the pigeonhole principle, that no injective function $f : A \to B$ can exist.

8. Let $A = \{1,2\}$ and $B = \{p,q,r\}$. Prove that no surjective function $f : A \to B$ can exist.

9. Let $f : Z \to Z$ be defined by $f(x) = 3x + 2$. Show that f is injective. Is f surjective when the co-domain is Z? Justify your answer.

10. Let $f : \{1,2,3\} \to \{1,2,3\}$ be defined by $f(1) = 2$, $f(2) = 3$, $f(3) = 1$. Show that f is bijective, and state whether it has an inverse function.