DISCRETE MATHEMATICS

Chapter 1: Mathematical Logic

Engineering Notes

1.1 Statement and Notation

A statement (also called a proposition) is a declarative sentence that has exactly one truth value — it is either TRUE or FALSE, but never both at the same time and never neither. Questions, commands, exclamations, and sentences containing variables whose value is unknown are not statements, because a truth value cannot be assigned to them.

Notation

- Statements are represented using lowercase letters such as p, q, r, s, ... called statement variables or propositional variables.
- The truth value of a statement is denoted by T (True) or 1, and F (False) or 0.
- A statement that cannot be broken down further is called an atomic or primary statement (e.g., "Delhi is the capital of India").
- A statement built by combining one or more atomic statements using connectives is called a compound or molecular statement (e.g., "It is raining and the ground is wet").

Examples:

- "7 is a prime number" — a statement, truth value T.
- "x + 5 = 8" — not a statement unless the value of x is fixed, since its truth value depends on x.
- "Please close the door" — not a statement, it is a command.
- "What time is it?" — not a statement, it is a question.

1.2 Connectives

Connectives are symbols/words used to combine one or more statements to form new (compound) statements. The truth value of the compound statement depends on the truth values of its component statements and the type of connective used.

Negation (¬p or ~p)

Negation reverses the truth value of a statement. If p is a statement, ¬p (read as "not p") is true when p is false, and false when p is true.

p	¬p
T	F
F	T

Conjunction (p ∧ q)

The conjunction of two statements p and q, written p ∧ q ("p and q"), is true only when both p and q are true; otherwise it is false.

p	q	p ∧ q
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction (p ∨ q)

The disjunction of p and q, written p ∨ q ("p or q"), is false only when both p and q are false; otherwise it is true. This is the inclusive OR. (The exclusive OR, written p ⊕q, is true only when exactly one of p, q is true.)

p	q	p ∨ q
T	T	T
T	F	T
F	T	T
F	F	F

Statement Formulas and Truth Tables

A statement formula is an expression built from statement variables, connectives, and parentheses, e.g., (p ∧ q) → ¬r. A truth table lists every possible combination of truth values of the variables and shows the resulting truth value of the formula for each combination. If a formula has n distinct statement variables, its truth table has 2^n rows.

- Truth tables are used to determine whether a formula is a tautology, contradiction, or contingency, and to check equivalence between two formulas.

Conditional (p → q)

The conditional (implication) p → q, read "if p then q", is false only when p is true and q is false; in every other case it is true. Here p is called the antecedent (hypothesis) and q is the consequent (conclusion).

p	q	p → q
T	T	T
T	F	F
F	T	T
F	F	T

- Converse of p → q is q → p.
- Inverse of p → q is ¬p → ¬q.
- Contrapositive of p → q is ¬q → ¬p, and it is logically equivalent to the original conditional.

Biconditional (p ↔ q)

The biconditional $p \leftrightarrow q$, read "p if and only if q", is true when p and q have the same truth value (both true or both false), and false otherwise.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

- $p \leftrightarrow q$ is logically equivalent to $(p \rightarrow q) \wedge (q \rightarrow p)$.

Well-Formed Formulas (WFF)

A well-formed formula (WFF) is a syntactically valid expression of mathematical logic, built according to the following rules:

1. Every statement variable (p, q, r, ...) and every truth value (T, F) is a WFF.
2. If A is a WFF, then $\neg A$ is a WFF.
3. If A and B are WFFs, then $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$, and $(A \leftrightarrow B)$ are WFFs.
4. A string is a WFF only if it can be obtained by a finite number of applications of rules 1–3.

Tautologies

A statement formula that is true for every possible truth value assignment of its variables is called a tautology (denoted T_0), e.g., $p \vee \neg p$. A formula that is false for every assignment is called a contradiction or fallacy (denoted F_0), e.g., $p \wedge \neg p$. A formula that is neither a tautology nor a contradiction (true for some, false for other assignments) is called a contingency.

Equivalence of Formulas

Two statement formulas A and B are said to be logically equivalent, written $A \Leftrightarrow B$ (or $A \equiv B$), if they have identical truth values for every possible assignment of truth values to their variables — equivalently, $A \leftrightarrow B$ is a tautology. Equivalence can be verified using truth tables or by algebraic manipulation using known laws.

Some standard equivalences (laws of logic):

- Idempotent Laws: $p \vee p \Leftrightarrow p$, $p \wedge p \Leftrightarrow p$
- Associative Laws: $(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$; $(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$
- Commutative Laws: $p \vee q \Leftrightarrow q \vee p$; $p \wedge q \Leftrightarrow q \wedge p$
- Distributive Laws: $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$; $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$
- Identity Laws: $p \vee F \Leftrightarrow p$, $p \wedge T \Leftrightarrow p$
- Domination Laws: $p \vee T \Leftrightarrow T$, $p \wedge F \Leftrightarrow F$
- Complement Laws: $p \vee \neg p \Leftrightarrow T$, $p \wedge \neg p \Leftrightarrow F$
- Double Negation: $\neg(\neg p) \Leftrightarrow p$
- De Morgan's Laws: $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$; $\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$
- Conditional as disjunction: $p \rightarrow q \Leftrightarrow \neg p \vee q$

Duality Law

Two formulas A and A* are called duals of each other if A* is obtained from A by replacing every $\vee$ with $\wedge$, every $\wedge$ with $\vee$, every T with F, and every F with T (statement variables remain unchanged). The Principle of Duality states that if $A \Leftrightarrow B$, then $A^* \Leftrightarrow B^*$ — i.e., the dual of one equivalence is also a valid equivalence. This law is very useful for quickly generating a second identity from a known one without re-deriving it.

Tautological Implications

A formula A is said to tautologically imply a formula B, written $A \Rightarrow B$, if $A \to B$ is a tautology, i.e., whenever A is true, B is necessarily true. Tautological implication is the logical basis of valid arguments and rules of inference; unlike equivalence ($\Leftrightarrow$), implication ($\Rightarrow$) works in one direction only. Standard implications include:

- $p \wedge q \Rightarrow p$ (Simplification)
- $p \Rightarrow p \vee q$ (Addition)
- $\neg p \Rightarrow p \to q$
- $q \Rightarrow p \to q$
- $\neg(p \to q) \Rightarrow p$
- $p \wedge (p \to q) \Rightarrow q$ (Modus Ponens)
- $\neg q \wedge (p \to q) \Rightarrow \neg p$ (Modus Tollens)
- $\neg p \wedge (p \vee q) \Rightarrow q$ (Disjunctive Syllogism)
- $(p \to q) \wedge (q \to r) \Rightarrow (p \to r)$ (Hypothetical Syllogism)

1.3 Normal Forms

A given statement formula can be expressed in several equivalent ways. To compare formulas easily and for use in circuit design, formulas are converted into standard (normal) forms using the connectives $\wedge$, $\vee$, and $\neg$ only.

Disjunctive Normal Form (DNF)

A formula is in Disjunctive Normal Form if it is expressed as a disjunction (OR) of one or more conjunctions (AND) of variables or their negations — i.e., a "sum of products", such as $(p \wedge q) \vee (\neg p \wedge r)$. Each conjunctive term is called a minterm. When every minterm contains all the variables of the formula (each appearing once, either plain or negated), the formula is said to be in Principal Disjunctive Normal Form (PDNF), also called the sum-of-minterms form.

Conjunctive Normal Form (CNF)

A formula is in Conjunctive Normal Form if it is expressed as a conjunction (AND) of one or more disjunctions (OR) of variables or their negations — i.e., a "product of sums", such as $(p \vee q) \wedge (\neg p \vee r)$. Each disjunctive term is called a maxterm. When every maxterm contains all the variables of the formula, the formula is in Principal Conjunctive Normal Form (PCNF), also called the product-of-maxterms form.

Steps to Obtain a Normal Form

5. Eliminate the conditional (→) and biconditional (↔) connectives using equivalences: $p \to q \Leftrightarrow \neg p \vee q$, and $p \leftrightarrow q \Leftrightarrow (p \wedge q) \vee (\neg p \wedge \neg q)$.
6. Move all negations inward so that they apply only to individual variables, using De Morgan's laws and double negation.
7. Apply the distributive laws repeatedly to obtain the required sum-of-products (for DNF) or product-of-sums (for CNF) structure.
8. For principal forms, expand every term so that it contains every variable of the formula, using the law $p \Leftrightarrow p \wedge (q \vee \neg q)$ (for PDNF) or $p \Leftrightarrow p \vee (q \wedge \neg q)$ (for PCNF), then apply the distributive law and remove duplicate terms.

1.4 The Theory of Inference for the Statement Calculus

The theory of inference deals with determining whether a conclusion logically follows from a given set of premises (hypotheses). An argument consisting of premises P1, P2, ..., Pn and a conclusion Q is said to be valid if Q is true whenever all the premises P1, P2, ..., Pn are true, i.e., if $(P1 \wedge P2 \wedge ... \wedge Pn) \Rightarrow Q$ is a tautology.

Validity Using Truth Tables

To test validity by truth table, construct a combined truth table for all premises and the conclusion. The argument is valid if, in every row where all the premises are true, the conclusion is also true. If there exists even one row where all premises are true but the conclusion is false, the argument is invalid. This method is reliable but becomes lengthy as the number of variables increases, since the table has 2^n rows.

Rules of Inference

Rather than constructing full truth tables, an argument can be proved valid step by step using standard rules of inference, applied to the premises to derive the conclusion:

Rule of Inference	Form	Description
Modus Ponens	$p, p \to q \vdash q$	If p is true and p implies q, then q is true.
Modus Tollens	$\neg q, p \to q \vdash \neg p$	If q is false and p implies q, then p is false.
Hypothetical Syllogism	$p \to q, q \to r \vdash p \to r$	Chaining two implications.
Disjunctive Syllogism	$p \vee q, \neg p \vdash q$	If one of two alternatives is ruled out, the other holds.
Addition	$p \vdash p \vee q$	A true statement implies its disjunction with anything.
Simplification	$p \wedge q \vdash p$	From a conjunction, either part can be inferred.
Conjunction	$p, q \vdash p \wedge q$	Two true statements can be combined.
Constructive Dilemma	$(p \to q) \wedge (r \to s), p \vee r \vdash q \vee s$	Combines two implications with a disjunction of antecedents.
Destructive Dilemma	$(p \to q) \wedge (r \to s), \neg q \vee \neg s \vdash \neg p \vee \neg r$	Combines two implications with a disjunction of negated consequents.

Consistency of Premises

A set of premises P1, P2, ..., Pn is said to be consistent if their conjunction $P1 \wedge P2 \wedge ... \wedge Pn$ can be true, i.e., there is at least one assignment of truth values to the variables for which all premises hold simultaneously (the conjunction is not a contradiction). If the conjunction of the premises is a contradiction (always false), the premises are inconsistent, and from an inconsistent set of premises, any conclusion whatsoever can be validly derived.

Indirect Method of Proof

Also called proof by contradiction, the indirect method is used when a direct derivation of the conclusion from the premises is difficult. The procedure is:

9. Assume the negation of the conclusion ($\neg Q$) to be an additional premise.
10. Using the rules of inference, derive a contradiction (a statement of the form $R \wedge \neg R$, i.e., a contradiction/fallacy) from the premises together with $\neg Q$.
11. Since assuming $\neg Q$ leads to a contradiction, $\neg Q$ cannot be true; hence Q must be true, and the original argument is proved valid.

1.5 Predicate Calculus

Statement (propositional) calculus cannot express statements involving variables and quantities, such as "every student passed the exam". Predicate calculus extends propositional logic to handle such statements by introducing predicates, variables, and quantifiers.

Rules of Precedence of Logical Operators

When a formula contains several connectives without parentheses, the order in which they are applied is fixed by the following precedence (from highest to lowest priority):

12. $\neg$ (Negation) — highest precedence
13. $\wedge$ (Conjunction)
14. $\vee$ (Disjunction)
15. $\rightarrow$ (Conditional)
16. $\leftrightarrow$ (Biconditional) — lowest precedence

For example, in the formula $\neg p \vee q \wedge r \rightarrow s$, negation is applied first, then conjunction, then disjunction, and finally the conditional, giving $((\neg p) \vee (q \wedge r)) \rightarrow s$. Parentheses may always be used to override the default precedence and remove ambiguity.

Predicate (Propositional) Functions

A predicate (or propositional function) is an expression containing one or more variables, defined over a certain domain, which becomes a statement (with a definite truth value) only when specific values from the domain are substituted for the variables. It is usually written as P(x), P(x, y), etc.

- Example: Let P(x): "x is an even number", defined over the domain of integers. P(4) is the statement "4 is an even number", which is true; P(7) is "7 is an even number", which is false.
- A predicate with n variables, P(x1, x2, ..., xn), is called an n-place predicate or an n-ary predicate.

- Predicates become full statements either by substituting particular values for the variables, or by quantifying the variables (using the universal quantifier ∀ meaning "for all", or the existential quantifier ∃ meaning "there exists").
- Example: $\forall x\, P(x)$ reads "for every x, P(x) holds"; $\exists x\, P(x)$ reads "there exists an x for which P(x) holds".

Predicate calculus thus provides the foundation for expressing and reasoning about mathematical statements that propositional logic alone cannot represent.

Multiple Choice Questions (MCQs)

1. Which of the following is a statement (proposition)?

a) Shut the window.
b) What is your name?
c) The sun rises in the east.
d) $x + 2 = 7$

2. The negation of the statement "It is raining" is:

a) It is sunny
b) It is not raining
c) It was raining
d) It may rain

3. The conjunction $p \wedge q$ is true when:

a) p is true only
b) q is true only
c) both p and q are true
d) either p or q is true

4. $p \vee q$ is false only when:

a) p is true, q is false
b) both p and q are false
c) p is false, q is true
d) both p and q are true

5. The conditional $p \rightarrow q$ is false only when:

a) p is false, q is true
b) p is true, q is true
c) p is true, q is false
d) both p and q are false

6. The contrapositive of $p \rightarrow q$ is:

a) $q \rightarrow p$
b) $\neg p \rightarrow \neg q$
c) $\neg q \rightarrow \neg p$
d) $\neg p \rightarrow q$

7. A formula that is true for every truth value assignment of its variables is called a:

a) Contradiction
b) Contingency
c) Tautology
d) Predicate

8. $p \wedge \neg p$ is an example of a:

a) Tautology
b) Contradiction
c) Contingency
d) Biconditional

9. According to De Morgan's Law, $\neg(p \vee q)$ is equivalent to:

a) $\neg p \vee \neg q$
b) $\neg p \wedge \neg q$

c) $p \wedge q$
d) $p \vee q$

10. The dual of $p \wedge (q \vee F)$ is:

a) $p \vee (q \wedge T)$
b) $p \wedge (q \vee T)$
c) $p \vee (q \wedge F)$
d) $p \wedge (q \wedge T)$

11. A formula expressed as a disjunction of conjunctions of variables/negations is in:

a) Conjunctive Normal Form
b) Disjunctive Normal Form
c) Principal form only
d) Predicate form

12. The rule of inference "p, $p \to q \vdash q$" is known as:

a) Modus Tollens
b) Modus Ponens
c) Hypothetical Syllogism
d) Simplification

13. A set of premises is called inconsistent if:

a) their conjunction is a tautology
b) their conjunction is a contradiction
c) they contain more than two variables
d) they cannot be written as a truth table

14. In the indirect method of proof, we begin by assuming:

a) the conclusion is true
b) the premises are false
c) the negation of the conclusion
d) a tautology

15. According to the standard precedence of logical operators, which connective is applied first in an unparenthesized formula?

a) $\to$ (Conditional)
b) $\vee$ (Disjunction)
c) $\wedge$ (Conjunction)
d) $\neg$ (Negation)

Answer Key

1-c, 2-b, 3-c, 4-b, 5-c, 6-c, 7-c, 8-b, 9-b, 10-a, 11-b, 12-b, 13-b, 14-c, 15-d

Broad / Long Answer Questions

1. Define a statement. Explain with suitable examples the difference between a statement and a sentence that is not a statement.

2. Explain the connectives negation, conjunction, and disjunction with their truth tables. What is the difference between inclusive OR and exclusive OR?

3. Define conditional and biconditional statements. Explain the converse, inverse, and contrapositive of a conditional statement with an example.

4. Define well-formed formula (WFF). State the rules used to construct a WFF and give two examples each of a valid and an invalid formula.

5. Define tautology, contradiction, and contingency. Using a truth table, show that $(p \to q) \lor (q \to p)$ is a tautology.

6. What is meant by equivalence of two statement formulas? State and prove any five standard laws of equivalence (e.g., De Morgan's laws, distributive laws).

7. State the Principle of Duality. Find the dual of the formula $(p \land q) \lor (\neg p \land F)$, and verify the duality law with a suitable example.

8. Obtain the Principal Disjunctive Normal Form (PDNF) and Principal Conjunctive Normal Form (PCNF) of the formula $(p \to q) \land (q \to r)$.

9. Explain the theory of inference. State and illustrate at least six standard rules of inference (Modus Ponens, Modus Tollens, Hypothetical Syllogism, Disjunctive Syllogism, Addition, Simplification) with one example each.

10. What is meant by consistency of premises? Explain the indirect method of proof with a suitable example, showing how a contradiction is derived from the negated conclusion.