# DISCRETE MATHEMATICS

## Chapter 5: Matrix Theory

*Study Notes for Engineering Students*

## 5.1 Elementary Transformation on a Matrix

### Elementary Transformations

An elementary transformation (or elementary operation) is an operation performed on the rows or columns of a matrix that changes its form without altering certain fundamental properties, such as its rank. Elementary transformations are of three kinds:

1. Interchange of any two rows (or columns), denoted $R_i \leftrightarrow R_j$ (or $C_i \leftrightarrow C_j$).
2. Multiplication of the elements of a row (or column) by a non-zero scalar k, denoted $R_i \rightarrow kR_i$, $k \neq 0$.
3. Addition to the elements of a row (or column) of k times the corresponding elements of another row (or column), denoted $R_i \rightarrow R_i + kR_j$.

Transformations performed on rows are called row transformations and are denoted by the prefix R; those performed on columns are called column transformations and are denoted by C.

### Equivalent Matrices

Two matrices A and B of the same order are said to be equivalent if one can be obtained from the other by applying a finite number of elementary transformations. This is written as $A \sim B$. Equivalent matrices always have the same order and, importantly, the same rank, since elementary transformations do not change the rank of a matrix.

### Sub-matrix of a Matrix

A matrix that is obtained from a given matrix by deleting any number of rows and/or any number of columns (not necessarily consecutive) is called a sub-matrix of the given matrix. For example, from a $3 \times 4$ matrix, deleting one row and two columns produces a $2 \times 2$ sub-matrix. Sub-matrices are used directly in the definition of rank through the concept of minors, since a minor of order r is simply the determinant of an $r \times r$ square sub-matrix.

### Rank of a Matrix

The rank of a matrix A, denoted by $\rho(A)$ or $r(A)$, is defined as the order of the largest non-vanishing (non-zero) minor of the matrix. Equivalently, the rank of a matrix is the maximum number of linearly independent rows (or, equivalently, linearly independent columns) present in the matrix.

Formally, a matrix A is said to be of rank r if it satisfies both of the following conditions:

- A has at least one minor of order r which is not equal to zero.
- Every minor of A of order $(r + 1)$, if any exist, is equal to zero.

A few useful observations about rank:

- The rank of a null (zero) matrix is defined to be 0.

- For a matrix of order $m \times n$, the rank can never exceed $\min(m, n)$.
- A non-singular square matrix of order n has rank exactly n (full rank); such a matrix is also called a matrix of full rank.
- The rank of a matrix remains unchanged under elementary row and column transformations.

## Echelon Form of a Matrix

A matrix is said to be in row echelon form if it satisfies the following conditions:

- Every row that consists entirely of zeros occurs below every row that has at least one non-zero entry.
- In each non-zero row, the first non-zero entry (called the leading entry or pivot) lies strictly to the right of the leading entry of the row above it, so that the number of leading zeros increases row by row, producing a staircase pattern.

Once a matrix has been reduced to echelon form using elementary row transformations, its rank is simply equal to the number of non-zero rows in that echelon form. This gives the most practical and widely used method of computing the rank of a matrix.

## Theorems on Rank (Statement Only)

4. The rank of a matrix is unaltered (invariant) by elementary transformations.
5. The rank of a matrix A is equal to the rank of its transpose $A^T$, i.e., $\rho(A) = \rho(A^T)$.
6. If A is a matrix of order $m \times n$, then $\rho(A) \leq \min(m, n)$.
7. The rank of the product of two matrices AB does not exceed the rank of either factor, i.e., $\rho(AB) \leq \min\{\rho(A), \rho(B)\}$.
8. Every matrix of rank r can be reduced, by a finite sequence of elementary transformations, to the normal form consisting of an identity block Ir together with zero blocks, written symbolically as [ Ir 0 ; 0 0 ].

## Evaluation of the Rank of a Matrix

In practice, the rank of a matrix is evaluated by reducing it to echelon form using elementary row operations and then counting the number of non-zero rows. The general procedure is as follows:

9. Use row operations to make the entry in the first column of the first row non-zero (interchange rows if necessary).
10. Use this leading entry to reduce all entries below it in the same column to zero, using $R_i \rightarrow R_i + kR_1$ type operations.
11. Repeat the process for the second row and second column onward, working diagonally down the matrix.
12. Continue until the matrix is in echelon form; the number of non-zero rows remaining is the rank of the matrix.

### *Solved Problem*

**Problem 1**

Find the rank of the matrix A = [ 1 2 3 ; 2 4 7 ; 3 6 10 ].

Solution: Apply $R_2 \rightarrow R_2 - 2R_1$ and $R_3 \rightarrow R_3 - 3R_1$:

$A \sim [\ 1\ 2\ 3\ ;\ 0\ 0\ 1\ ;\ 0\ 0\ 1\ ]$.

Now apply $R3 \rightarrow R3 - R2$:

$A \sim [\ 1\ 2\ 3\ ;\ 0\ 0\ 1\ ;\ 0\ 0\ 0\ ]$.

This is in echelon form with 2 non-zero rows. Hence, $\rho(A) = 2$.

## 5.2 Adjoint and Inverse of a Square Matrix

### Adjoint of a Square Matrix

Let $A = [a_{ij}]$ be a square matrix of order n. The cofactor $C_{ij}$ of the element $a_{ij}$ is defined as $C_{ij} = (-1)^{(i+j)} M_{ij}$, where $M_{ij}$ is the minor of $a_{ij}$ obtained by deleting the i-th row and j-th column of A. The matrix formed by replacing every element of A with its corresponding cofactor is called the cofactor matrix of A.

The adjoint (or adjugate) of A, denoted adj(A), is defined as the transpose of the cofactor matrix of A. That is, if $C = [C_{ij}]$ is the cofactor matrix of A, then $adj(A) = C^T$.

A fundamental property connecting a matrix and its adjoint is:

$$A \cdot adj(A) = adj(A) \cdot A = |A| \cdot I$$

where I is the identity matrix of the same order as A and |A| denotes the determinant of A. This identity is the basis for defining and computing the inverse of a matrix.

### Inverse of a Matrix

A square matrix A of order n is said to be invertible (or non-singular) if there exists a square matrix B of the same order such that AB = BA = I. The matrix B, when it exists, is called the inverse of A and is denoted $A^{-1}$.

Using the property $A \cdot adj(A) = |A| \cdot I$, the inverse of A can be expressed as:

$$A^{-1} = adj(A) / |A|, \quad provided\ |A| \neq 0$$

Thus, a square matrix possesses an inverse if and only if its determinant is non-zero, i.e., if and only if the matrix is non-singular. A matrix with $|A| = 0$ is called a singular matrix and does not possess an inverse.

### Uniqueness of the Inverse

The inverse of a matrix, whenever it exists, is unique. This can be shown as follows: suppose B and C are both inverses of A, so that AB = BA = I and AC = CA = I. Then:

$$B = B \cdot I = B \cdot (AC) = (BA) \cdot C = I \cdot C = C$$

Since B = C, the inverse of A, if it exists, must be unique.

### Theorems on Inverse of Matrices

13. The inverse of the inverse of a matrix returns the original matrix: $(A^{-1})^{-1} = A$.
14. For two invertible matrices A and B of the same order, the inverse of their product is the product of their inverses in reverse order: $(AB)^{-1} = B^{-1}A^{-1}$.
15. The inverse of the transpose of a matrix equals the transpose of its inverse: $(A^T)^{-1} = (A^{-1})^T$.
16. The determinant of the inverse of a matrix is the reciprocal of the determinant of the matrix: $|A^{-1}| = 1 / |A|$.

17. For a non-zero scalar k, $(kA)^{-1} = (1/k) A^{-1}$.

***Solved Problem***

**Problem 2**

Find the adjoint and inverse of A = [ 1 2 ; 3 4 ].

Solution: $|A| = (1)(4) - (2)(3) = 4 - 6 = -2$.

Cofactors: $C_{11} = 4$, $C_{12} = -3$, $C_{21} = -2$, $C_{22} = 1$.

Cofactor matrix = [ 4 −3 ; −2 1 ], so adj(A) = [ 4 −2 ; −3 1 ].

$A^{-1}$ = adj(A)/|A| = (1/−2) [ 4 −2 ; −3 1 ] = [ −2  1 ; 1.5  −0.5 ].

## 5.3 System of Simultaneous Linear Equations

A system of m linear equations in n unknowns $x_1, x_2, \ldots, x_n$ can be written in matrix form as AX = B, where A is the m × n coefficient matrix, X is the n × 1 column vector of unknowns, and B is the m × 1 column vector of constants on the right-hand side. The matrix [A | B], formed by appending B as an extra column to A, is called the augmented matrix of the system.

### Test of Consistency (Rouché–Capelli Condition)

A system of linear equations AX = B is said to be consistent if it has at least one solution, and inconsistent if it has no solution. The consistency of the system is tested by comparing the rank of the coefficient matrix A with the rank of the augmented matrix [A | B]:

- If $\rho(A) = \rho([A \mid B]) = n$ (the number of unknowns), the system is consistent and has a unique solution.
- If $\rho(A) = \rho([A \mid B]) = r < n$, the system is consistent but has infinitely many solutions, with $(n - r)$ unknowns that may be assigned arbitrary values.
- If $\rho(A) \neq \rho([A \mid B])$, the system is inconsistent and has no solution.

### Solution of n Linear Equations in n Unknowns

When the number of equations equals the number of unknowns (m = n) and the coefficient matrix A is non-singular ($|A| \neq 0$), the system AX = B has a unique solution, which can be obtained by either of the following methods:

- Matrix inverse method: $X = A^{-1}B$.
- Cramer's rule: $x_i = |A_i| / |A|$, where $A_i$ is the matrix formed by replacing the i-th column of A with the column B.
- Gaussian elimination: reducing the augmented matrix [A | B] to echelon form and solving by back substitution.

***Solved Problem***

**Problem 3**

Solve: $x + y = 3$, $2x - y = 0$.

Solution: Here A = [ 1 1 ; 2 −1 ], $|A| = -1 - 2 = -3 \neq 0$, so a unique solution exists.

By Cramer's rule: x = |[ 3 1 ; 0 −1 ]| / |A| = (−3 − 0)/(−3) = 1.

$y = |[\ 1\ 3\ ;\ 2\ 0\ ]| / |A| = (0 - 6)/(-3) = 2$.

Hence $x = 1$, $y = 2$.

### Solution of m Linear Equations in n Unknowns (m < n)

When there are fewer equations than unknowns ($m < n$), the system, if consistent, can never have a unique solution — it will always have infinitely many solutions. If $\rho(A) = \rho([A \mid B]) = r$, then $(n - r)$ of the unknowns can be chosen arbitrarily, and the remaining r unknowns are then expressed in terms of these arbitrary values.

**Problem 4 (m < n)**

Solve: $x + y + z = 6$, $x - y + 2z = 5$ (2 equations, 3 unknowns).

Here $\rho(A) = \rho([A \mid B]) = 2 < 3$, so the system is consistent with one arbitrary unknown.

Let $z = k$. Adding and subtracting the equations gives $x = (11 - 3k)/2$, $y = (1 + k)/2$, $z = k$, for any real k.

### Solution of m Linear Equations in n Unknowns (m > n)

When there are more equations than unknowns ($m > n$), the system is called an over-determined system. Such a system is generally inconsistent unless some of the equations are dependent on the others (i.e., some equations are linear combinations of the rest). Consistency is again checked using $\rho(A) = \rho([A \mid B])$; if this condition holds, the redundant equations may be discarded and the system solved using any n independent equations.

**Problem 5 (m > n)**

Test the consistency of: $x + y = 2$, $2x + 2y = 4$, $3x + y = 5$ (3 equations, 2 unknowns).

The first two equations are proportional (equation 2 = 2 × equation 1), so effectively there are only 2 independent equations in 2 unknowns.

Solving $x + y = 2$ and $3x + y = 5$ gives $x = 1.5$, $y = 0.5$, which also satisfies the second equation.

Since $\rho(A) = \rho([A \mid B]) = 2 = n$, the system is consistent with a unique solution $x = 1.5$, $y = 0.5$.

## 5.4 Eigenvalues and Eigenvectors

### Definition of Eigenvalues and Eigenvectors

Let A be a square matrix of order n. A scalar λ is called an eigenvalue (also called a characteristic value or latent root) of A if there exists a non-zero column vector X such that:

$$AX = \lambda X$$

The non-zero vector X satisfying this relation is called an eigenvector (also called a characteristic vector or latent vector) of A corresponding to the eigenvalue λ.

### Characteristic Equation

The relation $AX = \lambda X$ can be rewritten as $(A - \lambda I)X = 0$, which is a homogeneous system of linear equations in X. This system has a non-trivial (non-zero) solution X if and only if the coefficient matrix $(A - \lambda I)$ is singular, that is:

$$|A - \lambda I| = 0$$

This is called the characteristic equation of the matrix A. When expanded, it is a polynomial equation of degree n in $\lambda$, called the characteristic polynomial. The n roots of this equation (counted with multiplicity) are the eigenvalues of A.

## Relation between Characteristic Roots and Characteristic Vectors

Once the characteristic equation $|A - \lambda I| = 0$ is solved to obtain the eigenvalues $\lambda 1, \lambda 2, \ldots, \lambda n$, the corresponding eigenvector for each $\lambda i$ is obtained by substituting $\lambda i$ back into $(A - \lambda I)X = 0$ and solving this homogeneous system for X. Since the system is homogeneous, eigenvectors are never unique — any non-zero scalar multiple of an eigenvector is also an eigenvector for the same eigenvalue. Eigenvectors corresponding to distinct eigenvalues of a matrix are always linearly independent of one another.

## Nature of Characteristic Roots of Special Types of Matrices

- The eigenvalues of a symmetric matrix ($A = A^T$) are always real.
- The eigenvalues of a skew-symmetric matrix ($A = -A^T$) are either zero or purely imaginary.
- The eigenvalues of an orthogonal matrix ($AA^T = I$) always have modulus (absolute value) equal to 1.
- The eigenvalues of a Hermitian matrix are always real, and those of a skew-Hermitian matrix are either zero or purely imaginary.
- The eigenvalues of a triangular matrix (upper or lower) or a diagonal matrix are simply the entries lying on its principal diagonal.
- The eigenvalues of an idempotent matrix ($A^2 = A$) are either 0 or 1.
- If $\lambda$ is an eigenvalue of A, then $\lambda^k$ is an eigenvalue of $A^k$, and (if A is invertible) $1/\lambda$ is an eigenvalue of $A^{-1}$.

## Process of Finding Eigenvalues and Eigenvectors

18. Form the matrix $(A - \lambda I)$ by subtracting $\lambda$ from each diagonal element of A.
19. Set up and expand the characteristic equation $|A - \lambda I| = 0$ to obtain a polynomial in $\lambda$.
20. Solve this polynomial equation to find all the eigenvalues $\lambda 1, \lambda 2, \ldots, \lambda n$.
21. For each eigenvalue $\lambda i$, substitute it into $(A - \lambda i\ I)X = 0$ and solve the resulting homogeneous system to find the corresponding eigenvector Xi.

## Theorems on Eigenvalues and Eigenvectors

22. The sum of the eigenvalues of a matrix A equals the trace of A (the sum of its diagonal elements).
23. The product of the eigenvalues of a matrix A equals the determinant of A, i.e., $|A|$.
24. The eigenvalues of $A^T$ are the same as the eigenvalues of A.
25. If A is invertible, the eigenvalues of $A^{-1}$ are the reciprocals of the eigenvalues of A.
26. Cayley–Hamilton Theorem: every square matrix satisfies its own characteristic equation. This theorem is often used to compute higher powers of a matrix or its inverse without direct matrix multiplication.

***Solved Problem***

**Problem 6**

Find the eigenvalues and eigenvectors of A = [ 2 1 ; 1 2 ].

Characteristic equation: $|A - \lambda I| = (2-\lambda)^2 - 1 = 0 \Rightarrow \lambda^2 - 4\lambda + 3 = 0 \Rightarrow (\lambda-1)(\lambda-3) = 0$.

So the eigenvalues are $\lambda 1 = 1$ and $\lambda 2 = 3$.

For $\lambda 1 = 1$: $(A - I)X = 0$ gives $x + y = 0$, so an eigenvector is $X1 = (1, -1)^T$.

For $\lambda 2 = 3$: $(A - 3I)X = 0$ gives $-x + y = 0$, so an eigenvector is $X2 = (1, 1)^T$.

Check: sum of eigenvalues = 1 + 3 = 4 = trace(A); product = 1 × 3 = 3 = |A|.

## Multiple Choice Questions (MCQs)

Choose the correct option for each of the following 15 questions.

| Q. No. | Question and Options |
|---|---|
| 1 | **The rank of a null (zero) matrix is:**<br>(a) 0<br>(b) 1<br>(c) equal to its order<br>(d) undefined |
| 2 | **If A is a matrix of order m × n, then its rank can never exceed:**<br>(a) m + n<br>(b) min(m, n)<br>(c) max(m, n)<br>(d) mn |
| 3 | **Two matrices of the same order connected by a finite chain of elementary transformations are called:**<br>(a) Similar matrices<br>(b) Orthogonal matrices<br>(c) Equivalent matrices<br>(d) Conjugate matrices |
| 4 | **In a matrix reduced to echelon form, the rank equals:**<br>(a) The number of columns<br>(b) The number of zero rows<br>(c) The number of non-zero rows<br>(d) The order of the matrix |
| 5 | **The adjoint of a square matrix A is defined as:**<br>(a) The transpose of A<br>(b) The transpose of the cofactor matrix of A<br>(c) The inverse of A<br>(d) The determinant of A |
| 6 | **For a square matrix A, A · adj(A) equals:**<br>(a) I<br>(b) \|A\| · I<br>(c) $A^2$<br>(d) adj(A) · adj(A) |
| 7 | **A square matrix A possesses an inverse if and only if:**<br>(a) \|A\| = 0<br>(b) \|A\| ≠ 0<br>(c) A is symmetric |

| Q. No. | Question and Options |
|---|---|
| | (d) A is a null matrix |
| 8 | **If a matrix has an inverse, that inverse is:**<br>(a) Never unique<br>(b) Unique<br>(c) Equal to adj(A)<br>(d) Equal to A itself |
| 9 | **For invertible matrices A and B of the same order, $(AB)^{-1}$ equals:**<br>(a) $A^{-1}B^{-1}$<br>(b) $B^{-1}A^{-1}$<br>(c) $(BA)^{-1}$ only if AB = BA<br>(d) AB |
| 10 | **For the system AX = B, if ρ(A) = ρ([A\|B]) = n (number of unknowns), the system is:**<br>(a) Inconsistent<br>(b) Consistent with infinite solutions<br>(c) Consistent with a unique solution<br>(d) Consistent only if B = 0 |
| 11 | **If ρ(A) ≠ ρ([A \| B]) for a system AX = B, the system is:**<br>(a) Consistent with a unique solution<br>(b) Consistent with infinitely many solutions<br>(c) Inconsistent<br>(d) Homogeneous |
| 12 | **A system of m equations in n unknowns with m < n, if consistent, has:**<br>(a) A unique solution<br>(b) No solution<br>(c) Infinitely many solutions<br>(d) Exactly two solutions |
| 13 | **For a square matrix A, a non-zero vector X satisfying AX = λX for some scalar λ is called:**<br>(a) A cofactor of A<br>(b) An eigenvector of A<br>(c) A sub-matrix of A<br>(d) The adjoint of A |
| 14 | **The characteristic equation of a square matrix A is obtained from:**<br>(a) \|A\| = 0<br>(b) A − I = 0<br>(c) \|A − λI\| = 0<br>(d) AX = 0 |
| 15 | **The sum of the eigenvalues of a square matrix A is equal to:**<br>(a) \|A\|<br>(b) The trace of A<br>(c) The rank of A<br>(d) The order of A |

## Answer Key

**1-(a) 2-(b) 3-(c) 4-(c) 5-(b) 6-(b) 7-(b) 8-(b) 9-(b) 10-(c) 11-(c) 12-(c) 13-(b) 14-(c) 15-(b)**

## Broad / Long Answer Questions

1. Define equivalent matrices and sub-matrix of a matrix. Explain, with a suitable example, how a sub-matrix is used in defining the rank of a matrix.
2. Define the rank of a matrix. Reduce a given matrix of your choice to echelon form using elementary row transformations and hence determine its rank.
3. State (without proof) any four important theorems on the rank of a matrix, and explain their significance in the study of systems of linear equations.
4. Define the adjoint of a square matrix. Find the adjoint of a given $3 \times 3$ matrix and verify the relation $A \cdot \text{adj}(A) = |A| \cdot I$.
5. Prove that the inverse of a square matrix, when it exists, is unique.
6. State and prove any three theorems on the inverse of matrices, including the result $(AB)^{-1} = B^{-1}A^{-1}$.
7. State the test of consistency (Rouché–Capelli condition) for a system of simultaneous linear equations $AX = B$, and explain all three possible cases with the help of the ranks of $A$ and $[A \mid B]$.
8. Solve a system of $n$ linear equations in $n$ unknowns using (i) the matrix inverse method and (ii) Cramer's rule, illustrating both with the same example.
9. Discuss, with suitable examples, the solution of a system of $m$ linear equations in $n$ unknowns for the two cases $m < n$ and $m > n$.
10. Define the eigenvalues and eigenvectors of a square matrix. Explain the method of finding the characteristic equation, and find the eigenvalues and eigenvectors of a given $3 \times 3$ matrix.