DISCRETE MATHEMATICS

Chapter 3: Relation Between Two Sets

Engineering Notes

3.1 Relation Between Two Sets

In mathematics, a relation is used to describe how elements of one set are associated with elements of another set. Just as the Cartesian product of two sets collects every possible pairing of their elements, a relation picks out only those pairings that satisfy some particular condition.

Binary Relation as a Subset of Cartesian Product

Let A and B be two non-empty sets. A binary relation R from A to B is defined as any subset of the Cartesian product A × B, i.e., $R \subseteq A \times B$. If $(a, b) \in R$, we say that a is related to b under R, and we write a R b. If $(a, b) \notin R$, we write $a \not R b$.

- A is called the domain set and B is called the co-domain set of the relation R.
- The domain of R is the set of all first elements of the ordered pairs in R: $Dom(R) = \{a \in A \mid (a, b) \in R$ for some $b \in B\}$.
- The range of R is the set of all second elements of the ordered pairs in R: $Range(R) = \{b \in B \mid (a, b) \in R$ for some $a \in A\}$.
- Since R is a subset of A × B, and n(A × B) = n(A) × n(B), the total number of possible relations from A to B is $2^{(n(A) \times n(B))}$.
- If A = B, then $R \subseteq A \times A$ is called a relation on the set A (or an on the set relation).

Examples

Example 1: Let A = {1, 2, 3} and B = {2, 4, 6}. Define $R = \{(a, b) \mid a \in A, b \in B, b = 2a\}$. Then A × B has 9 ordered pairs, and R = {(1,2), (2,4), (3,6)} is a relation from A to B, since it is a subset of A × B.

Example 2: Let A = {1, 2, 3, 4}. Define a relation R on A by "a R b if a divides b". Then R = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)}, which is a subset of A × A.

Example 3: A relation can also be represented pictorially by an arrow diagram (an arrow drawn from each a to each related b) or by a matrix, where the entry in row a and column b is 1 if a R b, and 0 otherwise.

3.2 Reflexive, Symmetric and Transitive Relations

A relation on a set may possess certain special properties. Three of the most important such properties — reflexivity, symmetry, and transitivity — are used to classify relations and, together, define the important notion of an equivalence relation (Section 3.3).

Reflexive Relation

A relation R on a set A is said to be reflexive if every element of A is related to itself, i.e., $(a, a) \in R$ for every $a \in A$. Equivalently, R is reflexive if the diagonal relation $\{(a, a) \mid a \in A\}$ is contained in R.

- Example: On the set A = {1, 2, 3}, the relation R = {(1,1), (2,2), (3,3), (1,2)} is reflexive, since (1,1), (2,2), (3,3) are all present in R.
- Example: The relation "is less than or equal to" (≤) on the set of real numbers is reflexive, since $a \leq a$ for every real number a.
- Example: The relation "is strictly less than" (<) on the set of real numbers is not reflexive, since $a < a$ is false for every a.

Symmetric Relation

A relation R on a set A is said to be symmetric if, whenever a is related to b, b is also related to a, i.e., $(a, b) \in R \Rightarrow (b, a) \in R$ for all $a, b \in A$.

- Example: On the set A = {1, 2, 3}, the relation R = {(1,2), (2,1), (2,3), (3,2)} is symmetric, since for every pair present, its reverse pair is also present.
- Example: The relation "is a sibling of" on a set of people is symmetric, since if a is a sibling of b, then b is also a sibling of a.
- Example: The relation "is less than" (<) on real numbers is not symmetric, since $a < b$ does not imply $b < a$.

Transitive Relation

A relation R on a set A is said to be transitive if, whenever a is related to b and b is related to c, then a is also related to c, i.e., $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b, c \in A$.

- Example: On the set A = {1, 2, 3}, the relation R = {(1,2), (2,3), (1,3)} is transitive, since (1,2) and (2,3) are present and so is (1,3).
- Example: The relation "is an ancestor of" on a set of people is transitive, since if a is an ancestor of b and b is an ancestor of c, then a is an ancestor of c.
- Example: The relation "is a parent of" is not transitive, since if a is a parent of b and b is a parent of c, a is not a parent of c (a is a grandparent of c, not a parent).

Combined Example

Let A = {1, 2, 3, 4} and let R = {(1,1), (2,2), (3,3), (4,4), (1,2), (2,1)} be a relation on A. Checking each property: R is reflexive because $(1,1), (2,2), (3,3), (4,4) \in R$. R is symmetric because $(1,2) \in R$ and $(2,1) \in R$. R is transitive because the only pairs available (namely (1,2) and (2,1)) combine to give (1,1) and (2,2), both of which are already present in R. Hence R satisfies all three properties.

3.3 Equivalence Relation

A relation R on a set A is called an equivalence relation if it is simultaneously reflexive, symmetric, and transitive. Equivalence relations are extremely important because they formalize the idea of "sameness" or "equal-ness" between elements of a set with respect to some property, without requiring the elements to be literally identical.

Equivalence Classes

If R is an equivalence relation on a set A, then for each a ∈ A, the equivalence class of a, denoted [a], is the set of all elements of A that are related to a: [a] = {x ∈ A | x R a}. Every element of A belongs to exactly one equivalence class, and two equivalence classes are either identical or completely disjoint.

Examples

Example 1 (Equality): The relation "=" on any set A is an equivalence relation, since a = a (reflexive), a = b ⟹ b = a (symmetric), and a = b, b = c ⟹ a = c (transitive).

Example 2 (Congruence modulo n): On the set of integers Z, define a R b if and only if a ≡ b (mod n), i.e., (a − b) is divisible by n. This relation is reflexive (a − a = 0 is divisible by n), symmetric (if n divides a − b, it divides b − a), and transitive (if n divides a − b and b − c, it divides (a−b)+(b−c) = a−c). Hence congruence modulo n is an equivalence relation, and it partitions Z into n equivalence classes, called residue classes: [0], [1], [2], ..., [n−1].

Example 3 (Similar triangles): The relation "is similar to" on the set of all triangles in a plane is an equivalence relation, since every triangle is similar to itself, similarity is mutual, and similarity is transitive.

Example 4 (Parallel lines): The relation "is parallel to" on the set of all straight lines in a plane is an equivalence relation (taking every line to be parallel to itself).

Non-example: The relation "is perpendicular to" on the set of lines in a plane is symmetric but neither reflexive nor transitive, so it is not an equivalence relation.

3.4 Partition

A partition of a non-empty set A is a collection of non-empty subsets A1, A2, A3, ..., An (called blocks or cells) of A such that:

1. Every element of A belongs to exactly one of the subsets, i.e., A1 ∪ A2 ∪ ... ∪ An = A (the blocks cover A).
2. The subsets are pairwise disjoint, i.e., Ai ∩ Aj = ∅ for all i ≠ j (no element belongs to more than one block).
3. None of the subsets is empty, i.e., Ai ≠ ∅ for every i.

Relation Between Partition and Equivalence Relation

Partitions and equivalence relations are two ways of describing the same underlying structure on a set:

- Every equivalence relation R on a set A induces a partition of A, whose blocks are precisely the distinct equivalence classes of R.
- Conversely, every partition of a set A defines an equivalence relation on A, where two elements are related if and only if they lie in the same block of the partition.

This one-to-one correspondence between equivalence relations and partitions is one of the most important results in this topic.

Problems on Partition

Problem 1: Let A = {1, 2, 3, 4, 5, 6}. Show that the collection P = {{1, 2}, {3, 4, 5}, {6}} is a partition of A. Solution: The union {1,2} ∪ {3,4,5} ∪ {6} = {1,2,3,4,5,6} = A, so the blocks cover A. Also {1,2} ∩ {3,4,5} = ∅, {3,4,5} ∩ {6} =

$\emptyset$, and $\{1,2\} \cap \{6\} = \emptyset$, so the blocks are pairwise disjoint. None of the blocks is empty. Hence P is a partition of A.

Problem 2: Let A = Z (the set of integers) and let R be the relation "a R b if a and b leave the same remainder when divided by 3". Find the partition induced by R. Solution: R is an equivalence relation (congruence modulo 3), and it produces exactly three equivalence classes: $[0] = \{..., -6, -3, 0, 3, 6, ...\}$, $[1] = \{..., -5, -2, 1, 4, 7, ...\}$, and $[2] = \{..., -4, -1, 2, 5, 8, ...\}$. These three classes are pairwise disjoint, their union is Z, and none is empty, so $\{[0], [1], [2]\}$ is the partition of Z induced by R.

Problem 3: Let A = {1, 2, 3, 4}. Verify whether the collection {{1, 2}, {2, 3}, {4}} is a partition of A. Solution: The blocks {1,2} and {2,3} both contain the element 2, so they are not disjoint (their intersection is $\{2\} \neq \emptyset$). Since the pairwise-disjoint condition fails, this collection is not a partition of A.

Multiple Choice Questions (MCQs)

1. A binary relation R from set A to set B is defined as:

a) A subset of $A \cup B$
b) A subset of $A \times B$
c) A subset of $A \cap B$
d) A subset of $P(A)$

2. If n(A) = 3 and n(B) = 4, the total number of possible relations from A to B is:

a) 12
b) 2^7
c) 2^12
d) 7^2

3. The domain of a relation $R \subseteq A \times B$ is:

a) The set of all second elements of pairs in R
b) The set B itself
c) The set of all first elements of pairs in R
d) Always equal to A

4. A relation R on a set A is reflexive if:

a) $(a,b) \in R \Rightarrow (b,a) \in R$
b) $(a,a) \in R$ for every $a \in A$
c) $(a,b),(b,c) \in R \Rightarrow (a,c) \in R$
d) $R = \emptyset$

5. The relation "<" (strictly less than) on the set of real numbers is:

a) Reflexive
b) Symmetric
c) Not reflexive
d) An equivalence relation

6. A relation R on a set A is symmetric if:

a) $(a,a) \in R$ for all a
b) $(a,b) \in R \Rightarrow (b,a) \in R$
c) $(a,b),(b,c) \in R \Rightarrow (a,c) \in R$
d) R has no elements

7. The relation "is a sibling of" on a set of people is an example of a relation that is:

a) Reflexive only
b) Symmetric
c) Antisymmetric
d) Not symmetric

8. A relation R on a set A is transitive if:

a) $(a,b) \in R \Rightarrow (b,a) \in R$
b) $(a,a) \in R$ for all a
c) $(a,b),(b,c) \in R \Rightarrow (a,c) \in R$
d) $R = A \times A$

9. Which combination of properties defines an equivalence relation?

a) Reflexive and symmetric only
b) Symmetric and transitive only

c) Reflexive, symmetric, and transitive
d) Transitive only

10. The relation "congruence modulo n" on the set of integers is:

a) Only reflexive
b) Only symmetric
c) An equivalence relation
d) Not transitive

11. For an equivalence relation R on set A, the equivalence class [a] is defined as:

a) $\{x \in A \mid x\ R\ a\}$
b) {a}
c) A × A
d) The complement of a in A

12. Two distinct equivalence classes of the same equivalence relation are always:

a) Equal
b) Overlapping
c) Disjoint
d) Empty

13. A partition of a set A requires that the blocks be:

a) Overlapping and non-empty
b) Pairwise disjoint, non-empty, and cover A
c) Equal in size
d) A single block equal to A

14. Every equivalence relation on a set A corresponds to:

a) A single subset of A
b) A partition of A into equivalence classes
c) The empty relation
d) A symmetric matrix only

15. Which of the following collections is NOT a partition of A = {1,2,3,4}?

a) {{1,2},{3,4}}
b) {{1},{2,3},{4}}
c) {{1,2},{2,3},{4}}
d) {{1,2,3,4}}

Answer Key

1-b, 2-c, 3-c, 4-b, 5-c, 6-b, 7-b, 8-c, 9-c, 10-c, 11-a, 12-c, 13-b, 14-b, 15-c

Broad / Long Answer Questions

1. Define a binary relation from set A to set B. Explain how a relation is a subset of the Cartesian product A × B, with a suitable example.

2. Define the domain and range of a relation. If A = {1,2,3} and B = {2,4,6} and R = {(a,b) | b = 2a}, find R, its domain, and its range.

3. Define a reflexive relation with an example. Show that the relation "≤" on the set of real numbers is reflexive but the relation "<" is not.

4. Define a symmetric relation with an example. Determine whether the relation "is a sibling of" on a set of people is symmetric, giving reasons.

5. Define a transitive relation with an example. Check whether R = {(1,2),(2,3),(1,3)} on A = {1,2,3} is transitive, giving reasons.

6. Define an equivalence relation. Prove that the relation "congruence modulo n" on the set of integers is an equivalence relation.

7. Define equivalence class. If R is the relation of congruence modulo 4 on the set of integers, list the distinct equivalence classes formed.

8. Define a partition of a set. State the three conditions a collection of subsets must satisfy to form a partition, with a suitable example.

9. State and explain the relationship between an equivalence relation and a partition of a set, with an example showing how one determines the other.

10. Let A = {1,2,3,4,5,6}. Verify whether each of the following is a partition of A: (i) {{1,2},{3,4,5},{6}}, (ii) {{1,2},{2,3},{4,5,6}}. Justify your answers.