DISCRETE MATHEMATICS

Chapter 2: Set Theory

Engineering Notes

2.1 Concept of Sets

A set is a well-defined collection of distinct objects, considered as a single entity. The objects that make up a set are called its elements or members. A set is said to be well-defined if, given any object, it is possible to decide definitely whether that object belongs to the set or not.

Notation

- Sets are usually denoted by capital letters A, B, C, ... and their elements by small letters a, b, c, ...
- If x is an element of set A, we write $x \in A$ (read "x belongs to A"); if x is not an element of A, we write $x \notin A$.
- Roster (Tabular) method: elements are listed within braces, e.g., A = {1, 2, 3, 4}.
- Set-builder (Rule) method: a set is described by a property its elements satisfy, e.g., A = {x | x is a natural number and x < 5}.
- The number of distinct elements in a set A is called its cardinality, denoted n(A) or |A|.

Subset

A set A is called a subset of a set B, written $A \subseteq B$, if every element of A is also an element of B. If $A \subseteq B$ and $A \neq B$ (i.e., at least one element of B is not in A), then A is called a proper subset of B, written $A \subset B$. Every set is a subset of itself, and the empty set is a subset of every set.

Superset

If A is a subset of B ($A \subseteq B$), then B is called a superset of A, written $B \supseteq A$. In other words, B contains all the elements of A and possibly more.

Empty Set

A set that contains no elements is called the empty set, null set, or void set, denoted by $\emptyset$ or { }. Its cardinality is 0, i.e., $n(\emptyset) = 0$. The empty set is a subset of every set, including itself.

Universal Set

The universal set, denoted U, is the set that contains all the objects or elements under consideration for a particular discussion or problem. Every other set discussed in that context is a subset of the universal set.

Examples

- Let A = {2, 4, 6, 8}. Here $4 \in A$ but $5 \notin A$, and n(A) = 4.
- Let A = {1, 2} and B = {1, 2, 3, 4}. Then $A \subseteq B$ and, since $A \neq B$, also $A \subset B$; equivalently $B \supseteq A$.
- If U = {1, 2, 3, ..., 10} is the universal set, then A = {2, 4, 6} and B = {1, 3, 5} are both subsets of U.
- The set of all students in a class who scored more than 100 marks out of 100 is an empty set, $\emptyset$.

2.2 Operations on Sets

New sets can be formed from given sets by performing certain operations on them. The most common operations are union, intersection, complementation, difference, and symmetric difference.

Union ($A \cup B$)

The union of two sets A and B, written $A \cup B$, is the set of all elements that belong to A or to B or to both. Symbolically, $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$.

- Example: If $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$, then $A \cup B = \{1, 2, 3, 4, 5\}$.

Intersection ($A \cap B$)

The intersection of two sets A and B, written $A \cap B$, is the set of all elements that belong to both A and B. Symbolically, $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$. If $A \cap B = \emptyset$, the sets A and B are said to be disjoint.

- Example: If $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$, then $A \cap B = \{3\}$.

Complementation (A' or A^c)

The complement of a set A with respect to a universal set U, written A' (or A^c or $\bar{A}$), is the set of all elements of U that are not in A. Symbolically, $A' = U - A = \{x \mid x \in U \text{ and } x \notin A\}$.

- Example: If $U = \{1, 2, 3, ..., 10\}$ and $A = \{2, 4, 6, 8, 10\}$, then $A' = \{1, 3, 5, 7, 9\}$.

Difference ($A - B$)

The difference of two sets A and B, written $A - B$ (or $A \setminus B$), is the set of all elements that belong to A but not to B. Symbolically, $A - B = \{x \mid x \in A \text{ and } x \notin B\}$. In general, $A - B \neq B - A$.

- Example: If $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$, then $A - B = \{1, 2\}$ and $B - A = \{4, 5\}$.

Symmetric Difference ($A \triangle B$)

The symmetric difference of two sets A and B, written $A \triangle B$ (or $A \oplus B$), is the set of elements that belong to exactly one of A or B, but not to both. Symbolically, $A \triangle B = (A - B) \cup (B - A) = (A \cup B) - (A \cap B)$.

- Example: If $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$, then $A \triangle B = \{1, 2, 4, 5\}$.

Problems Relating to Simple Set Identities

Set identities are laws that hold for all sets, and are proved either by showing mutual subset containment (element-wise proof) or with the help of Venn diagrams. The important set identities are:

- Idempotent Laws: $A \cup A = A$, $A \cap A = A$
- Commutative Laws: $A \cup B = B \cup A$, $A \cap B = B \cap A$
- Associative Laws: $(A \cup B) \cup C = A \cup (B \cup C)$, $(A \cap B) \cap C = A \cap (B \cap C)$
- Distributive Laws: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- Identity Laws: $A \cup \emptyset = A$, $A \cap U = A$
- Domination Laws: $A \cup U = U$, $A \cap \emptyset = \emptyset$
- Complement Laws: $A \cup A' = U$, $A \cap A' = \emptyset$, $(A')' = A$
- De Morgan's Laws: $(A \cup B)' = A' \cap B'$, $(A \cap B)' = A' \cup B'$

- Absorption Laws: $A \cup (A \cap B) = A$, $A \cap (A \cup B) = A$

Sample problem: Prove that $A - (B \cup C) = (A - B) \cap (A - C)$. Proof (element-wise): Let $x \in A - (B \cup C)$. Then $x \in A$ and $x \notin (B \cup C)$, i.e., $x \in A$ and ($x \notin B$ and $x \notin C$). This gives ($x \in A$ and $x \notin B$) and ($x \in A$ and $x \notin C$), i.e., $x \in (A - B)$ and $x \in (A - C)$, so $x \in (A - B) \cap (A - C)$. The reverse containment follows similarly, so the two sets are equal.

2.3 Power Set and Cartesian Product

Power Set

The power set of a set A, denoted P(A), is the set of all possible subsets of A, including the empty set $\emptyset$ and the set A itself. If a set A has n elements ($n(A) = n$), then its power set has exactly 2^n elements, i.e., $n(P(A)) = 2^n$.

- Example: If $A = \{1, 2\}$, then $P(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$, and $n(P(A)) = 2^2 = 4$.
- Example: If $A = \{a, b, c\}$, then $n(P(A)) = 2^3 = 8$, and $P(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$.

Cartesian Product of a Finite Number of Sets

The Cartesian product of two sets A and B, denoted $A \times B$, is the set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$. Symbolically, $A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$. In general, $A \times B \neq B \times A$ unless $A = B$ or one of them is empty.

This idea extends to any finite number of sets. The Cartesian product of n sets A1, A2, ..., An, denoted $A1 \times A2 \times ... \times An$, is the set of all ordered n-tuples (a1, a2, ..., an) such that $ai \in Ai$ for each i = 1, 2, ..., n.

- If $n(A) = m$ and $n(B) = n$, then $n(A \times B) = m \times n$.
- More generally, $n(A1 \times A2 \times ... \times An) = n(A1) \times n(A2) \times ... \times n(An)$.

Simple Problems

Example: Let $A = \{1, 2\}$ and $B = \{a, b, c\}$. Then $A \times B = \{(1,a), (1,b), (1,c), (2,a), (2,b), (2,c)\}$, so $n(A \times B) = 2 \times 3 = 6$.

Example: Let $A = \{x, y\}$ and $B = \{1, 2\}$. Find $A \times B$ and $B \times A$, and verify that they are different. Here $A \times B = \{(x,1), (x,2), (y,1), (y,2)\}$ while $B \times A = \{(1,x), (1,y), (2,x), (2,y)\}$. Both have 4 ordered pairs, but the pairs themselves are different, confirming $A \times B \neq B \times A$.

2.4 Cardinality of a Set

The cardinality (or cardinal number) of a set A, denoted n(A) or |A|, is the number of distinct elements present in the set. Cardinality is used to compare the "size" of sets and to count the outcomes of set operations.

- $n(\emptyset) = 0$ — the empty set has cardinality zero.
- If A and B are disjoint sets ($A \cap B = \emptyset$), then $n(A \cup B) = n(A) + n(B)$.
- For any two sets A and B (Principle of Inclusion–Exclusion): $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.
- For three sets A, B, C: $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$.
- $n(A - B) = n(A) - n(A \cap B)$.

Example: In a class of 50 students, 30 play cricket, 20 play football, and 10 play both games. Number of students playing at least one game = $n(C \cup F) = n(C) + n(F) - n(C \cap F) = 30 + 20 - 10 = 40$.

2.5 Finite and Infinite Sets

Finite Set

A set is called a finite set if it contains a definite (countable) number of elements, i.e., the process of counting its elements terminates after a certain stage. The cardinality of a finite set is a non-negative integer.

- Example: A = {1, 2, 3, 4, 5} is a finite set with n(A) = 5.
- Example: The set of letters in the English alphabet is a finite set with 26 elements.
- The empty set ∅ is also considered a finite set, with cardinality 0.

Infinite Set

A set is called an infinite set if it is not finite, i.e., the elements of the set cannot be counted completely because the counting process never terminates.

- Example: The set of natural numbers N = {1, 2, 3, 4, ...} is an infinite set.
- Example: The set of all points on a line segment is an infinite set.
- Infinite sets are further classified as countably infinite (elements can be put in one-to-one correspondence with natural numbers, e.g., the set of integers) and uncountably infinite (cannot be so listed, e.g., the set of real numbers).

Multiple Choice Questions (MCQs)

1. Which of the following best defines a set?

a) Any random group of numbers
b) A well-defined collection of distinct objects
c) A list that can repeat elements
d) A single object

2. If A = {1, 2, 3} and B = {1, 2, 3, 4, 5}, then:

a) $A \supseteq B$
b) $A \subset B$
c) $B \subset A$
d) A = B

3. The empty set is:

a) A subset of every set
b) A superset of every set
c) Not a subset of any set
d) Equal to the universal set

4. If U = {1,2,...,10} and A = {1,2,3}, then A′ is:

a) {1,2,3}
b) {4,5,6,7,8,9,10}
c) $\emptyset$
d) U

5. If A = {1,2,3} and B = {2,3,4}, then $A \cap B$ is:

a) {1,2,3,4}
b) {2,3}
c) {1,4}
d) $\emptyset$

6. If A = {1,2,3} and B = {2,3,4}, then A − B is:

a) {1}
b) {4}
c) {2,3}
d) {1,2,3,4}

7. The symmetric difference $A \triangle B$ is defined as:

a) $A \cap B$
b) $(A - B) \cup (B - A)$
c) $A \cup B$
d) $(A \cup B) \cap (A \cap B)$

8. According to De Morgan's Law, $(A \cup B)'$ equals:

a) $A' \cup B'$
b) $A' \cap B'$
c) $A \cap B$
d) $A' - B'$

9. Two sets A and B are called disjoint if:

a) $A \cup B = \emptyset$
b) $A \cap B = \emptyset$

c) A = B

d) A − B = ∅

10. If a set A has 4 elements, the number of elements in its power set P(A) is:

a) 4

b) 8

c) 16

d) 12

11. If A = {1, 2} and B = {a, b, c}, then n(A × B) is:

a) 5

b) 6

c) 8

d) 2

12. In general, A × B and B × A are:

a) Always equal

b) Equal only if A = B

c) Never defined

d) Always disjoint

13. If n(A) = 20, n(B) = 15, n(A ∩ B) = 5, then n(A ∪B) is:

a) 40

b) 35

c) 30

d) 25

14. Which of the following is an example of an infinite set?

a) The set of days in a week

b) The set of natural numbers

c) The set of vowels in English

d) The empty set

15. The cardinality of the empty set is:

a) 0

b) 1

c) Undefined

d) Infinite

Answer Key

1-b, 2-b, 3-a, 4-b, 5-b, 6-a, 7-b, 8-b, 9-b, 10-c, 11-b, 12-b, 13-c, 14-b, 15-a

Broad / Long Answer Questions

1. Define a set. Explain the roster method and set-builder method of representing a set with suitable examples.

2. Define subset, proper subset, and superset. Distinguish between the empty set and the universal set with examples.

3. Define the union and intersection of two sets. If A = {1,2,3,4} and B = {3,4,5,6}, find $A \cup B$ and $A \cap B$, and represent them using Venn diagrams.

4. Define the complement, difference, and symmetric difference of sets with examples. Show that $A \triangle B = (A \cup B) - (A \cap B)$.

5. State and prove (using the element-wise method) De Morgan's Laws for set theory: $(A \cup B)' = A' \cap B'$ and $(A \cap B)' = A' \cup B'$.

6. Prove the distributive law $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ using an element-wise argument, and verify it with a suitable example.

7. Define the power set of a set. If A = {a, b, c, d}, find P(A) and verify that $n(P(A)) = 2^n$.

8. Define the Cartesian product of two sets. If A = {1, 2, 3} and B = {x, y}, find $A \times B$ and $B \times A$, and show that $A \times B \neq B \times A$.

9. Define cardinality of a set. State the Principle of Inclusion–Exclusion for two and three sets, and solve: In a survey of 100 people, 60 like tea, 50 like coffee, and 30 like both. Find how many like at least one of the two drinks.

10. Distinguish between finite and infinite sets with suitable examples. Briefly explain the difference between countably infinite and uncountably infinite sets.