

MATHEMATICS 1

Chapter: Binomial Theorem

Important Exam-Oriented Questions with Solutions

Section A: Important Formulas (Quick Revision)

- **Binomial theorem (n a positive integer):** $(a + b)^n = \sum {}^nC_r a^{n-r} b^r$, where $r = 0, 1, 2, \dots, n$.
- **Number of terms:** $(n + 1)$ in the expansion of $(a + b)^n$.
- **General term:** $T_{r+1} = {}^nC_r a^{n-r} b^r$.
- **Middle term:** if n is even, one middle term $T_{n/2+1}$; if n is odd, two middle terms $T_{(n+1)/2}$ and $T_{(n+3)/2}$.
- **r-th term from the end** = $(n - r + 2)$ -th term from the beginning.
- **Ratio of consecutive terms:** $T_{r+1} / T_r = [(n - r + 1) / r] \times (b / a)$.
- **Term independent of x:** put the power of x in the general term equal to 0 and find r .
- **Sum of coefficients:** put every variable equal to 1.
- **Properties:** ${}^nC_r = {}^nC_{n-r}$; ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$; if ${}^nC_x = {}^nC_y$ then $x = y$ or $x + y = n$; $r \cdot {}^nC_r = n \cdot {}^{n-1}C_{r-1}$.
- **Sum of binomial coefficients:** $C_0 + C_1 + \dots + C_n = 2^n$; $C_0 + C_2 + C_4 + \dots = C_1 + C_3 + C_5 + \dots = 2^{n-1}$.
- **Any index (negative or fraction), $|x| < 1$:** $(1 + x)^n = 1 + nx + [n(n-1)/2!] x^2 + [n(n-1)(n-2)/3!] x^3 + \dots$ (infinite terms).
- **General term (any index):** $T_{r+1} = [n(n-1)(n-2)\dots(n-r+1) / r!] x^r$.
- **Standard series:** $(1 + x)^{-1} = 1 - x + x^2 - x^3 + \dots$; $(1 - x)^{-1} = 1 + x + x^2 + x^3 + \dots$; $(1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \dots$; $(1 - x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots$; $(1 - x)^{-3} = 1 + 3x + 6x^2 + 10x^3 + \dots$
- **Approximation:** when x is very small, $(1 + x)^n \approx 1 + nx$.

Section B: Short Answer Questions (2 Marks)

Q1. State the binomial theorem for a positive integer n . How many terms are there in the expansion?

Solution:

1. For any positive integer n : $(a + b)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1} b + {}^nC_2 a^{n-2} b^2 + \dots + {}^nC_n b^n$.
2. Compact form: $(a + b)^n = \sum {}^nC_r a^{n-r} b^r$, $r = 0$ to n .
3. Power of a decreases from n to 0, power of b increases from 0 to n , and the sum of the powers in every term is n .

Ans. The expansion has $(n + 1)$ terms.

Q2. Expand $(2a - b)^4$.**Solution:**

1. Take $x = 2a$ and $y = -b$ in $(x + y)^4$. Coefficients from Pascal's triangle: 1, 4, 6, 4, 1.
2. $(2a)^4 = 16a^4$
3. $4(2a)^3(-b) = -32a^3b$
4. $6(2a)^2(-b)^2 = 24a^2b^2$
1. $4(2a)(-b)^3 = -8ab^3$
2. $(-b)^4 = b^4$

Ans. $(2a - b)^4 = 16a^4 - 32a^3b + 24a^2b^2 - 8ab^3 + b^4$.**Q3. Write the general term and the middle term(s) of $(a + b)^n$.****Solution:**

1. General term: $T_{r+1} = {}^nC_r a^{n-r} b^r$.
2. If n is even, the number of terms $(n + 1)$ is odd, so there is one middle term: $T_{n/2 + 1}$.
3. If n is odd, $(n + 1)$ is even, so there are two middle terms: $T_{(n+1)/2}$ and $T_{(n+3)/2}$.

Ans. General term = ${}^nC_r a^{n-r} b^r$; middle term rule as given above.**Q4. Find the 6th term in the expansion of $(2x + 3)^8$.****Solution:**

1. Here $n = 8$, first term $a = 2x$, second term $b = 3$. For the 6th term, $r + 1 = 6$, so $r = 5$.
2. $T_6 = {}^8C_5 (2x)^3 (3)^5$
3. ${}^8C_5 = 56$, $(2x)^3 = 8x^3$, $3^5 = 243$
4. $T_6 = 56 \times 8 \times 243 \times x^3 = 108864 x^3$

Ans. $T_6 = 108864 x^3$.**Q5. Find the middle term of $(x + 1/x)^{10}$.****Solution:**

1. $n = 10$ is even, so there is one middle term: $T_{n/2 + 1} = T_6$, with $r = 5$.
2. $T_6 = {}^{10}C_5 x^5 (1/x)^5$
3. $= 252 \times x^5 \times x^{-5} = 252$

Ans. Middle term = 252.**Q6. Find the sum of the coefficients in the expansion of $(2x - 3y)^{10}$.****Solution:**

1. Sum of coefficients is obtained by putting $x = 1$ and $y = 1$.
2. Sum = $(2 - 3)^{10} = (-1)^{10} = 1$

Ans. Sum of coefficients = 1.

Q7. Using the binomial theorem, evaluate $(99)^3$.

Solution:

1. Write $99 = 100 - 1$.
2. $(100 - 1)^3 = 100^3 - 3(100)^2(1) + 3(100)(1)^2 - 1^3$
3. $= 1000000 - 30000 + 300 - 1$

Ans. $(99)^3 = 970299$.

Q8. Expand $(1 + x)^{-2}$ up to the term containing x^3 .

Solution:

1. Here $n = -2$ (negative index), so the expansion is infinite and valid for $|x| < 1$.
2. Formula: $1 + nx + [n(n-1)/2!]x^2 + [n(n-1)(n-2)/3!]x^3$
3. Coefficient of x^2 : $(-2)(-3)/2 = 3$
4. Coefficient of x^3 : $(-2)(-3)(-4)/6 = -4$

Ans. $(1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \dots$

Q9. Find the ratio of the coefficients of x^p and x^q in the expansion of $(1 + x)^{p+q}$.

Solution:

1. Coefficient of $x^p = {}^{p+q}C_p = (p+q)! / (p! q!)$
2. Coefficient of $x^q = {}^{p+q}C_q = (p+q)! / (q! p!)$
3. Both are equal.

Ans. Ratio = 1 : 1.

Q10. If ${}^nC_8 = {}^nC_{12}$, find n and hence find nC_2 .

Solution:

1. If ${}^nC_x = {}^nC_y$, then $x = y$ or $x + y = n$. Here $8 \neq 12$, so $n = 8 + 12 = 20$.
2. ${}^{20}C_2 = (20 \times 19) / (2 \times 1) = 190$

Ans. $n = 20$ and ${}^nC_2 = 190$.

Q11. Find the number of terms in the expansion of $(x + y)^8 + (x - y)^8$.

Solution:

1. In the sum, terms with odd powers of y cancel, and terms with even powers of y get doubled.
2. $(x + y)^8 + (x - y)^8 = 2[x^8 + {}^8C_2x^6y^2 + {}^8C_4x^4y^4 + {}^8C_6x^2y^6 + y^8]$
3. Terms left: those with $r = 0, 2, 4, 6, 8$.

Ans. Number of terms = 5.

Section C: Medium Answer Questions (4 Marks)

Q12. Expand $(x + 2/x)^5$.

Solution:

1. General term: $T_{r+1} = {}^5C_r x^{5-r} (2/x)^r = {}^5C_r 2^r x^{5-2r}$.
2. $r = 0$: $1 \times 1 \times x^5 = x^5$
3. $r = 1$: $5 \times 2 \times x^3 = 10x^3$
4. $r = 2$: $10 \times 4 \times x = 40x$
5. $r = 3$: $10 \times 8 \times x^{-1} = 80/x$
6. $r = 4$: $5 \times 16 \times x^{-3} = 80/x^3$
7. $r = 5$: $1 \times 32 \times x^{-5} = 32/x^5$

Ans. $(x + 2/x)^5 = x^5 + 10x^3 + 40x + 80/x + 80/x^3 + 32/x^5$.

Q13. Find the term independent of x in the expansion of $(x^2 - 1/x)^9$.

Solution:

1. $T_{r+1} = {}^9C_r (x^2)^{9-r} (-1/x)^r = {}^9C_r (-1)^r x^{18-3r}$.
2. For the term independent of x , the power of x must be 0: $18 - 3r = 0$, so $r = 6$.
3. $T_7 = {}^9C_6 (-1)^6 = 84 \times 1$

Ans. Term independent of $x = 84$ (the 7th term).

Q14. Find the coefficient of x^{10} in the expansion of $(x^2 + 3/x)^{11}$.

Solution:

1. $T_{r+1} = {}^{11}C_r (x^2)^{11-r} (3/x)^r = {}^{11}C_r 3^r x^{22-3r}$.
2. Put $22 - 3r = 10$, so $r = 4$.
3. Coefficient = ${}^{11}C_4 \times 3^4 = 330 \times 81$

Ans. Coefficient of $x^{10} = 26730$.

Q15. Find the middle terms of the expansion of $(3x - x^3/6)^9$.

Solution:

1. $n = 9$ is odd, so there are two middle terms: T_5 and T_6 .
2. $T_5 = {}^9C_4 (3x)^5 (-x^3/6)^4 = 126 \times 243x^5 \times x^{12}/1296 = (126 \times 3/16) x^{17} = (189/8) x^{17}$
3. $T_6 = {}^9C_5 (3x)^4 (-x^3/6)^5 = 126 \times 81x^4 \times (-x^{15}/7776) = -(126/96) x^{19} = -(21/16) x^{19}$

Ans. Middle terms are $(189/8) x^{17}$ and $-(21/16) x^{19}$.

Q16. Find the 4th term from the end in the expansion of $(2x - 1/x^2)^{10}$.

Solution:

1. The r -th term from the end is the $(n - r + 2)$ -th term from the beginning.
2. Here $n = 10$, $r = 4$, so we need the $(10 - 4 + 2) = 8$ th term, i.e. T_8 with $r = 7$.
3. $T_8 = {}^{10}C_7 (2x)^3 (-1/x^2)^7 = 120 \times 8x^3 \times (-x^{-14})$

Ans. 4th term from the end = $-960 x^{-11}$.

Q17. If the coefficients of the 5th, 6th and 7th terms in the expansion of $(1 + x)^n$ are in A.P., find n.

Solution:

1. Coefficients: ${}^nC_4, {}^nC_5, {}^nC_6$. In an A.P., $2 {}^nC_5 = {}^nC_4 + {}^nC_6$.
2. Divide throughout by nC_5 : $2 = {}^nC_4/{}^nC_5 + {}^nC_6/{}^nC_5$.
3. Use ${}^nC_{r-1}/{}^nC_r = r/(n-r+1)$ and ${}^nC_{r+1}/{}^nC_r = (n-r)/(r+1)$: $2 = 5/(n-4) + (n-5)/6$.
4. Multiply by $6(n-4)$: $12(n-4) = 30 + (n-5)(n-4)$.
5. $12n - 48 = n^2 - 9n + 50$, so $n^2 - 21n + 98 = 0$.
6. $(n-7)(n-14) = 0$

Ans. $n = 7$ or $n = 14$.

Q18. Evaluate $(\sqrt{2} + 1)^5 + (\sqrt{2} - 1)^5$.

Solution:

1. Sum of $(a+b)^5 + (a-b)^5 = 2[{}^5C_0 a^5 + {}^5C_2 a^3 b^2 + {}^5C_4 a b^4]$, with $a = \sqrt{2}$ and $b = 1$.
2. $a^5 = 4\sqrt{2}$, $a^3 = 2\sqrt{2}$, $a = \sqrt{2}$
3. $= 2[4\sqrt{2} + 10(2\sqrt{2}) + 5\sqrt{2}] = 2[4\sqrt{2} + 20\sqrt{2} + 5\sqrt{2}] = 2 \times 29\sqrt{2}$

Ans. $(\sqrt{2} + 1)^5 + (\sqrt{2} - 1)^5 = 58\sqrt{2}$.

Q19. Find the value of a if the coefficients of x^2 and x^3 in the expansion of $(3 + ax)^9$ are equal.

Solution:

1. $T_{r+1} = {}^9C_r 3^{9-r} (ax)^r$.
2. Coefficient of x^2 ($r = 2$): ${}^9C_2 3^7 a^2 = 36 \times 3^7 a^2$.
3. Coefficient of x^3 ($r = 3$): ${}^9C_3 3^6 a^3 = 84 \times 3^6 a^3$.
4. Equate: $36 \times 3^7 a^2 = 84 \times 3^6 a^3$. Cancel $3^6 a^2$ ($a \neq 0$): $108 = 84a$.

Ans. $a = 9/7$.

Q20. Using the binomial theorem, find $(1.01)^5$ correct to 4 decimal places.

Solution:

1. $(1 + 0.01)^5 = 1 + 5(0.01) + 10(0.01)^2 + 10(0.01)^3 + \dots$
2. $= 1 + 0.05 + 0.001 + 0.00001 + \dots$
3. The remaining terms are too small to affect the 4th decimal place.

Ans. $(1.01)^5 \approx 1.0510$.

Q21. Prove that $C_0 + C_2 + C_4 + \dots = C_1 + C_3 + C_5 + \dots = 2^{n-1}$.

Solution:

1. Start with $(1 + x)^n = C_0 + C_1 x + C_2 x^2 + \dots + C_n x^n$.
2. Put $x = 1$: $2^n = C_0 + C_1 + C_2 + C_3 + \dots \dots (i)$

3. Put $x = -1$: $0 = C_0 - C_1 + C_2 - C_3 + \dots \dots$ (ii)
4. From (ii): $C_0 + C_2 + C_4 + \dots = C_1 + C_3 + C_5 + \dots \dots$ (iii)
5. Let each side of (iii) be S . Then (i) gives $S + S = 2^n$, so $S = 2^{n-1}$.

Ans. Hence proved: $C_0 + C_2 + \dots = C_1 + C_3 + \dots = 2^{n-1}$.

Section D: Long Answer Questions (6 Marks)

Q22. State and prove the binomial theorem for a positive integer n using mathematical induction.

Solution:

1. **Statement:** $(a + b)^n = \sum {}^nC_r a^{n-r} b^r$, $r = 0$ to n , for every positive integer n .
2. **Step 1 ($n = 1$):** LHS = $a + b$. RHS = ${}^1C_0 a + {}^1C_1 b = a + b$. So the result is true for $n = 1$.
3. **Step 2 (assume for $n = k$):** $(a + b)^k = \sum {}^kC_r a^{k-r} b^r$.
4. **Step 3 (prove for $n = k + 1$):** $(a + b)^{k+1} = (a + b)(a + b)^k = a(a + b)^k + b(a + b)^k$.
5. Expanding: $(a + b)^{k+1} = \sum {}^kC_r a^{k-r+1} b^r + \sum {}^kC_r a^{k-r} b^{r+1}$.
6. Collect the terms containing $a^{k+1-r} b^r$. Their coefficient is ${}^kC_r + {}^kC_{r-1}$.
7. By Pascal's rule, ${}^kC_r + {}^kC_{r-1} = {}^{k+1}C_r$.
8. Therefore $(a + b)^{k+1} = \sum {}^{k+1}C_r a^{k+1-r} b^r$, $r = 0$ to $k + 1$.
9. **Conclusion:** The result is true for $n = 1$, and true for $k + 1$ whenever it is true for k . By induction it is true for all positive integers n .

Ans. Binomial theorem proved.

Q23. Expand $(1 + x)^{1/2}$ up to x^3 and hence find the value of $\sqrt{1.02}$ approximately.

Solution:

1. Here $n = 1/2$, valid for $|x| < 1$.
2. Coefficient of x^2 : $(1/2)(-1/2)/2! = -1/8$
3. Coefficient of x^3 : $(1/2)(-1/2)(-3/2)/3! = (3/8)/6 = 1/16$
4. So $(1 + x)^{1/2} = 1 + x/2 - x^2/8 + x^3/16 - \dots$
5. Put $x = 0.02$ (so $1 + x = 1.02$): $x/2 = 0.01$, $x^2/8 = 0.0004/8 = 0.00005$, $x^3/16 = 0.000008/16 = 0.0000005$.
6. $\sqrt{1.02} \approx 1 + 0.01 - 0.00005 + 0.0000005 = 1.0099505$

Ans. $\sqrt{1.02} \approx 1.00995$.

Q24. Find the approximate value of $(998)^{1/3}$ correct to 4 decimal places.

Solution:

1. Write $998 = 1000 - 2 = 1000(1 - 0.002)$. So $(998)^{1/3} = 10(1 - 0.002)^{1/3}$.
2. Expand $(1 - x)^{1/3} = 1 - x/3 + [(1/3)(-2/3)/2!]x^2 - \dots = 1 - x/3 - x^2/9 - \dots$
3. Put $x = 0.002$: $x/3 = 0.000666667$ and $x^2/9 = 0.000000444$.

4. $(1 - 0.002)^{1/3} \approx 1 - 0.000666667 - 0.000000444 = 0.999332889$

5. Multiply by 10: 9.99332889

Ans. $(998)^{1/3} \approx 9.9933$.

Q25. Expand $(1 - 2x)^{-3}$ up to x^3 . Also find its general term and the coefficient of x^5 .

Solution:

1. Use $(1 - y)^{-3} = 1 + 3y + 6y^2 + 10y^3 + \dots$ with $y = 2x$.
2. $(1 - 2x)^{-3} = 1 + 3(2x) + 6(2x)^2 + 10(2x)^3 + \dots = 1 + 6x + 24x^2 + 80x^3 + \dots$
3. In $(1 - y)^{-3}$, the coefficient of y^r is $(r + 1)(r + 2)/2$. So the general term is $T_{r+1} = [(r + 1)(r + 2)/2] 2^r x^r$.
4. For x^5 , $r = 5$: coefficient $= (6 \times 7/2) \times 2^5 = 21 \times 32$

Ans. Series: $1 + 6x + 24x^2 + 80x^3 + \dots$; coefficient of $x^5 = 672$.

Q26. Prove that $C_1 + 2C_2 + 3C_3 + \dots + nC_n = n \cdot 2^{n-1}$.

Solution:

1. The general term of the LHS is $r \cdot {}^nC_r$.
2. $r \cdot {}^nC_r = r \cdot n! / [r!(n - r)!] = n \cdot (n - 1)! / [(r - 1)!(n - r)!] = n \cdot {}^{n-1}C_{r-1}$
3. So LHS $= n [{}^{n-1}C_0 + {}^{n-1}C_1 + {}^{n-1}C_2 + \dots + {}^{n-1}C_{n-1}]$.
4. The sum of all binomial coefficients of $(1 + x)^{n-1}$ is 2^{n-1} .
5. LHS $= n \cdot 2^{n-1}$.
6. Check with $n = 3$: $3 + 2(3) + 3(1) = 12 = 3 \times 2^2$.

Ans. Hence proved: $C_1 + 2C_2 + \dots + nC_n = n \cdot 2^{n-1}$.

Q27. Three consecutive coefficients in the expansion of $(1 + x)^n$ are in the ratio 1 : 7 : 42. Find n.

Find n.

Solution:

1. Let the coefficients be ${}^nC_{r-1}$, nC_r , ${}^nC_{r+1}$.
2. First ratio: ${}^nC_r / {}^nC_{r-1} = (n - r + 1)/r = 7$, so $n - r + 1 = 7r$, i.e. $n = 8r - 1 \dots$ (i)
3. Second ratio: ${}^nC_{r+1} / {}^nC_r = (n - r)/(r + 1) = 42/7 = 6$, so $n - r = 6r + 6$, i.e. $n = 7r + 6 \dots$ (ii)
4. From (i) and (ii): $8r - 1 = 7r + 6$, so $r = 7$.
5. Substitute in (ii): $n = 7(7) + 6 = 55$.
6. Check: ${}^{55}C_7 / {}^{55}C_6 = 49/7 = 7$ and ${}^{55}C_8 / {}^{55}C_7 = 48/8 = 6 \checkmark$

Ans. $n = 55$ (and $r = 7$).

Section E: Practice Questions with Answers

Solve these on your own and check your answers.

1. Expand $(1 + 2x)^4$. **Ans:** $1 + 8x + 24x^2 + 32x^3 + 16x^4$
2. Find the middle term of $(2x + y)^6$. **Ans:** $160x^3y^3$

3. Find the term independent of x in $(x + 1/x)^8$. **Ans:** 70
4. Evaluate $(11)^4$ using the binomial theorem. **Ans:** 14641
5. Find the value of $(0.99)^5$ approximately. **Ans:** 0.95099
6. Find the coefficient of x^4 in $(1 + x)^{-2}$. **Ans:** 5
7. Find the sum of the coefficients in $(3x - 2)^5$. **Ans:** 1
8. Expand $(1 - x)^{-2}$ up to x^3 . **Ans:** $1 + 2x + 3x^2 + 4x^3 + \dots$

Poly Notes Hub