

MATHEMATICS 1

Chapter: Binomial Theorem

Broad (Long Answer) Questions with Complete Solutions

Part 1: Terms and Coefficients

Q1. Find the term independent of x in the expansion of $(2x - 1/(3x^2))^9$.

Solution:

- Here $n = 9$, first term $= 2x$, second term $= -1/(3x^2)$.
- General term: $T_{r+1} = {}^9C_r (2x)^{9-r} (-1/(3x^2))^r$.
- Collect constants and powers of x separately: $T_{r+1} = {}^9C_r \cdot 2^{9-r} \cdot (-1)^r / 3^r \cdot x^{9-r-2r} = {}^9C_r \cdot 2^{9-r} \cdot (-1)^r / 3^r \cdot x^{9-3r}$.
- For the term independent of x , put $9 - 3r = 0$, so $r = 3$.
- It is the 4th term: $T_4 = {}^9C_3 \cdot 2^6 \cdot (-1)^3 / 3^3$.
- $T_4 = 84 \times 64 \times (-1) / 27 = -5376/27 = -1792/9$.

Ans. Term independent of $x = -1792/9$.

Q2. Find the coefficient of x^5 in the expansion of $(1 + 2x)^6 (1 - x)^7$.

Solution:

- Expand each bracket separately using the binomial theorem.
- $(1 + 2x)^6 = 1 + 12x + 60x^2 + 160x^3 + 240x^4 + 192x^5 + 64x^6$
- $(1 - x)^7 = 1 - 7x + 21x^2 - 35x^3 + 35x^4 - 21x^5 + 7x^6 - x^7$
- x^5 in the product is obtained by multiplying x^i from the first bracket with x^{5-i} from the second, for $i = 0$ to 5 :
- $(1)(-21) + (12)(35) + (60)(-35) + (160)(21) + (240)(-7) + (192)(1)$
- $= -21 + 420 - 2100 + 3360 - 1680 + 192 = 171$.

Ans. Coefficient of $x^5 = 171$.

Q3. Show that the middle term in the expansion of $(1 + x)^{2n}$ is $[1 \cdot 3 \cdot 5 \dots (2n - 1) / n!] 2^n x^n$.

Solution:

- The expansion of $(1 + x)^{2n}$ has $(2n + 1)$ terms, which is odd, so there is exactly one middle term: T_{n+1} .
- $T_{n+1} = {}^{2n}C_n x^n = [(2n)! / (n! n!)] x^n$.
- Split $(2n)!$ into odd and even factors: $(2n)! = [1 \cdot 3 \cdot 5 \dots (2n - 1)] \times [2 \cdot 4 \cdot 6 \dots 2n]$.
- Take 2 common from each even factor: $2 \cdot 4 \cdot 6 \dots 2n = 2^n (1 \cdot 2 \cdot 3 \dots n) = 2^n n!$.
- So $(2n)! = 2^n n! [1 \cdot 3 \cdot 5 \dots (2n - 1)]$.
- Substitute: $T_{n+1} = 2^n n! [1 \cdot 3 \cdot 5 \dots (2n - 1)] x^n / (n! n!) = [1 \cdot 3 \cdot 5 \dots (2n - 1) / n!] 2^n x^n$.

Ans. Hence proved.

Q4. Find the middle terms in the expansion of $(x/3 + 3y)^7$.

Solution:

1. $n = 7$ is odd, so there are two middle terms: $T_{(7+1)/2} = T_4$ and $T_{(7+3)/2} = T_5$.
2. $T_4 = {}^7C_3 (x/3)^4 (3y)^3 = 35 \times (x^4/81) \times 27y^3 = (35 \times 27/81) x^4 y^3 = (35/3) x^4 y^3$.
3. $T_5 = {}^7C_4 (x/3)^3 (3y)^4 = 35 \times (x^3/27) \times 81y^4 = (35 \times 81/27) x^3 y^4 = 105 x^3 y^4$.

Ans. Middle terms: $(35/3) x^4 y^3$ and $105 x^3 y^4$.

Q5. The coefficients of three consecutive terms in the expansion of $(1 + x)^n$ are in the ratio 1 : 3 : 5. Find n and the position of these terms.

Solution:

1. Let the three coefficients be ${}^nC_{r-1}$, nC_r , ${}^nC_{r+1}$.
2. Use ${}^nC_r / {}^nC_{r-1} = (n - r + 1)/r$.
3. First ratio: $(n - r + 1)/r = 3$, so $n - r + 1 = 3r$, i.e. $n = 4r - 1$... (i)
4. Use ${}^nC_{r+1} / {}^nC_r = (n - r)/(r + 1)$.
5. Second ratio: $(n - r)/(r + 1) = 5/3$, so $3n - 3r = 5r + 5$, i.e. $3n = 8r + 5$... (ii)
6. Substitute (i) in (ii): $3(4r - 1) = 8r + 5$, so $12r - 3 = 8r + 5$, so $r = 2$.
7. Then $n = 4(2) - 1 = 7$.
8. Check: ${}^7C_1 : {}^7C_2 : {}^7C_3 = 7 : 21 : 35 = 1 : 3 : 5$ ✓

Ans. $n = 7$. The terms are the 2nd, 3rd and 4th terms.

Q6. The 2nd, 3rd and 4th terms in the expansion of $(a + x)^n$ are 240, 720 and 1080 respectively. Find a , x and n .

Solution:

1. $T_2 = {}^nC_1 a^{n-1} x = 240$... (i)
2. $T_3 = {}^nC_2 a^{n-2} x^2 = 720$... (ii)
3. $T_4 = {}^nC_3 a^{n-3} x^3 = 1080$... (iii)
4. Divide (ii) by (i): $[(n - 1)/2] (x/a) = 3$... (iv)
5. Divide (iii) by (ii): $[(n - 2)/3] (x/a) = 3/2$... (v)
6. Divide (iv) by (v): $3(n - 1) / [2(n - 2)] = 2$, so $3n - 3 = 4n - 8$, so $n = 5$.
7. Put $n = 5$ in (iv): $2(x/a) = 3$, so $x = 3a/2$.
8. Put $n = 5$ and $x = 3a/2$ in (i): $5a^4 \times (3a/2) = 240$, so $a^5 = 32$, so $a = 2$.
9. Then $x = 3(2)/2 = 3$.
10. Check: $T_2 = 5 \times 16 \times 3 = 240$, $T_3 = 10 \times 8 \times 9 = 720$, $T_4 = 10 \times 4 \times 27 = 1080$ ✓

Ans. $a = 2$, $x = 3$, $n = 5$.

Part 2: Properties of Binomial Coefficients and Applications

Q7. Prove that $C_0^2 + C_1^2 + C_2^2 + \dots + C_n^2 = (2n)! / (n!)^2$.

Solution:

1. We have $(1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n \dots$ (i)
2. Also $(x+1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \dots + C_n \dots$ (ii), because ${}^nC_r = {}^nC_{n-r}$.
3. Multiply (i) and (ii): $(1+x)^n(x+1)^n = (1+x)^{2n}$.
4. Coefficient of x^n on the right side $= {}^{2n}C_n = (2n)! / (n!)^2$.
5. On the left side, x^n is formed by pairing C_rx^r from (i) with C_rx^{n-r} from (ii), for $r = 0$ to n .
6. So the coefficient of x^n on the left $= C_0C_0 + C_1C_1 + \dots + C_nC_n = C_0^2 + C_1^2 + \dots + C_n^2$.
7. Equating both coefficients gives the result. Check with $n = 2$: $1 + 4 + 1 = 6 = 4!/(2!)^2 \checkmark$

Ans. $C_0^2 + C_1^2 + \dots + C_n^2 = (2n)! / (n!)^2$. Hence proved.

Q8. Prove that $C_0 + C_1/2 + C_2/3 + \dots + C_n/(n+1) = (2^{n+1} - 1)/(n+1)$.

Solution:

1. The general term of the left side is ${}^nC_r / (r+1)$.
2. ${}^nC_r / (r+1) = n! / [r!(n-r)!(r+1)] = n! / [(r+1)!(n-r)!]$
3. Multiply and divide by $(n+1)$: $= (n+1)! / [(n+1)(r+1)!(n-r)!] = {}^{n+1}C_{r+1} / (n+1)$.
4. Take $r = 0, 1, 2, \dots, n$. Sum $= [{}^{n+1}C_1 + {}^{n+1}C_2 + \dots + {}^{n+1}C_{n+1}] / (n+1)$.
5. The sum of all binomial coefficients of $(1+x)^{n+1}$ is 2^{n+1} , and ${}^{n+1}C_0 = 1$ is missing from the bracket.
6. So the bracket $= 2^{n+1} - 1$.
7. Check with $n = 2$: $1 + 1 + 1/3 = 7/3$ and $(8-1)/3 = 7/3 \checkmark$

Ans. Left side $= (2^{n+1} - 1)/(n+1)$. Hence proved.

Q9. Using the binomial theorem, show that $9^n - 8n - 1$ is divisible by 64 for every positive integer n .

Solution:

1. Write $9 = 1 + 8$, so $9^n = (1+8)^n$.
2. Expand: $(1+8)^n = 1 + {}^nC_1(8) + {}^nC_2(8)^2 + {}^nC_3(8)^3 + \dots + 8^n$.
3. $= 1 + 8n + 64 {}^nC_2 + 64 \cdot 8 \cdot {}^nC_3 + \dots$
4. Every term from the third onwards contains $8^2 = 64$. Take 64 common: $9^n = 1 + 8n + 64[{}^nC_2 + 8 \cdot {}^nC_3 + 8^2 \cdot {}^nC_4 + \dots]$.
5. So $9^n - 8n - 1 = 64 \times [{}^nC_2 + 8 \cdot {}^nC_3 + 8^2 \cdot {}^nC_4 + \dots]$.
6. The bracket is a whole number, so the expression is a multiple of 64.

Ans. $9^n - 8n - 1$ is divisible by 64. Hence proved.

Q10. Find the remainder when 7^{103} is divided by 25, using the binomial theorem.

Solution:

1. Write $7^{103} = 7 \times 7^{102} = 7 \times (7^2)^{51} = 7 \times (49)^{51} = 7 \times (50 - 1)^{51}$.
2. Expand: $(50 - 1)^{51} = 50^{51} - {}^{51}C_1 50^{50} + \dots + {}^{51}C_{50} (50)(-1)^{50} + (-1)^{51}$.
3. Every term except the last contains the factor 50, hence is a multiple of 25. So $(50 - 1)^{51} = 25k - 1$ for some integer k.
4. $7^{103} = 7(25k - 1) = 175k - 7$.
5. Write $-7 = -25 + 18$: $7^{103} = 175k - 25 + 18 = 25(7k - 1) + 18$.
6. Check: $7^4 = 2401$ leaves remainder 1 on division by 25. $103 = 4 \times 25 + 3$, so 7^{103} leaves the same remainder as $7^3 = 343$, which is 18 ✓

Ans. Remainder = 18.

Part 3: Negative and Fractional Index, Approximations

Q11. Expand $(1 + x)^{-1/2}$ up to the term in x^3 . Hence find the value of $1/\sqrt{1.04}$ correct to 5 decimal places.

Solution:

1. Here $n = -1/2$, and the expansion is valid for $|x| < 1$.
2. Coefficient of x^2 : $n(n-1)/2! = (-1/2)(-3/2)/2 = 3/8$.
3. Coefficient of x^3 : $n(n-1)(n-2)/3! = (-1/2)(-3/2)(-5/2)/6 = -5/16$.
4. $(1 + x)^{-1/2} = 1 - x/2 + (3/8)x^2 - (5/16)x^3 + \dots$
5. Now $1/\sqrt{1.04} = (1 + 0.04)^{-1/2}$, so put $x = 0.04$.
6. $x/2 = 0.02$; $(3/8)x^2 = (3/8)(0.0016) = 0.0006$; $(5/16)x^3 = (5/16)(0.00064) = 0.0002$.
7. $1/\sqrt{1.04} \approx 1 - 0.02 + 0.0006 - 0.0002 = 0.9804$

Ans. $(1 + x)^{-1/2} = 1 - x/2 + (3/8)x^2 - (5/16)x^3 + \dots$; $1/\sqrt{1.04} \approx 0.9804$.

Q12. Find the approximate value of $(127)^{1/3}$ correct to 4 decimal places.

Solution:

1. Choose the nearest perfect cube: 125. Write $127 = 125 + 2 = 125(1 + 2/125) = 125(1 + 0.016)$.
2. So $(127)^{1/3} = 5(1 + 0.016)^{1/3}$.
3. Expand $(1 + x)^{1/3} = 1 + x/3 + [(1/3)(-2/3)/2!]x^2 + [(1/3)(-2/3)(-5/3)/3!]x^3 + \dots = 1 + x/3 - x^2/9 + 5x^3/81 - \dots$
4. Put $x = 0.016$: $x/3 = 0.0053333$; $x^2/9 = 0.000256/9 = 0.0000284$; $5x^3/81 = 0.00000025$.
5. $(1.016)^{1/3} \approx 1 + 0.0053333 - 0.0000284 + 0.0000003 = 1.0053052$.
6. Multiply by 5: $5 \times 1.0053052 = 5.026526$.

Ans. $(127)^{1/3} \approx 5.0265$.

Q13. Expand $(3 + 2x)^{-1}$ in ascending powers of x up to x^3 . State the range of x for which the expansion is valid, and find the coefficient of x^4 .

Solution:

1. Take 3 common from the bracket: $(3 + 2x)^{-1} = 3^{-1}(1 + 2x/3)^{-1} = (1/3)(1 + y)^{-1}$, where $y = 2x/3$.
2. Use $(1 + y)^{-1} = 1 - y + y^2 - y^3 + y^4 - \dots$
3. $(1/3)[1 - 2x/3 + 4x^2/9 - 8x^3/27 + \dots]$
4. $= 1/3 - 2x/9 + 4x^2/27 - 8x^3/81 + \dots$
5. Validity: the expansion of $(1 + y)^{-1}$ is valid for $|y| < 1$, so $|2x/3| < 1$, i.e. $|x| < 3/2$.
6. General term: $(1/3)(-1)^r(2x/3)^r$. For $r = 4$: $(1/3)(2/3)^4 = (1/3)(16/81) = 16/243$.

Ans. $(3 + 2x)^{-1} = 1/3 - 2x/9 + 4x^2/27 - 8x^3/81 + \dots$, valid for $|x| < 3/2$; coefficient of $x^4 = 16/243$.

Q14. Expand $(1 + x)/(1 - x)^2$ up to x^3 and find the coefficient of x^r .

Solution:

1. Write the expression as $(1 + x)(1 - x)^{-2}$.
2. Standard series (valid for $|x| < 1$): $(1 - x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots + (r + 1)x^r + \dots$
3. Multiply by $(1 + x)$: $(1 + x)(1 + 2x + 3x^2 + 4x^3 + \dots) = (1 + 2x + 3x^2 + 4x^3 + \dots) + (x + 2x^2 + 3x^3 + \dots)$
4. Add like powers: $1 + 3x + 5x^2 + 7x^3 + \dots$
5. Coefficient of x^r : from the first bracket $(r + 1)$, from the second bracket r . Total $= (r + 1) + r = 2r + 1$.

Ans. $(1 + x)/(1 - x)^2 = 1 + 3x + 5x^2 + 7x^3 + \dots$; coefficient of $x^r = 2r + 1$.