

MATHEMATICS 1

Chapter: Binomial Theorem

Broad (Long Answer) Questions with Complete Solutions: Set 2

Part 1: Terms, Coefficients and Greatest Term

Q1. Find the coefficient of x^{-3} in the expansion of $(x - 2/x^2)^{12}$.

Solution:

- Here $n = 12$, first term = x , second term = $-2/x^2$.
- General term: $T_{r+1} = {}^{12}C_r x^{12-r} (-2/x^2)^r = {}^{12}C_r (-2)^r x^{12-r-2r} = {}^{12}C_r (-2)^r x^{12-3r}$.
- For the required term, $12 - 3r = -3$, so $3r = 15$, so $r = 5$.
- It is the 6th term: $T_6 = {}^{12}C_5 (-2)^5 x^{-3}$.
- ${}^{12}C_5 = 792$ and $(-2)^5 = -32$, so the coefficient = $792 \times (-32) = -25344$.

Ans. Coefficient of $x^{-3} = -25344$.

Q2. Find the numerically greatest term in the expansion of $(3 + 2x)^9$ when $x = 1$.

Solution:

- Here $a = 3$, $b = 2x = 2$, $n = 9$. The general term is $T_{r+1} = {}^9C_r 3^{9-r} 2^r$.
- Ratio of consecutive terms: $T_{r+1} / T_r = [(n - r + 1)/r] \times (b/a) = [(10 - r)/r] \times (2/3)$.
- $T_{r+1} > T_r$ when $(10 - r)(2/3) > r$, i.e. $20 - 2r > 3r$, i.e. $r < 4$.
- So the terms keep increasing up to T_4 ($r = 1, 2, 3$). For $r = 4$ the ratio = $(6/4)(2/3) = 1$, so $T_5 = T_4$. For $r > 4$ the ratio is less than 1, so the terms decrease.
- Hence T_4 and T_5 are equal and greatest.
- $T_4 = {}^9C_3 3^6 2^3 = 84 \times 729 \times 8 = 489888$.
- $T_5 = {}^9C_4 3^5 2^4 = 126 \times 243 \times 16 = 489888 \checkmark$

Ans. The 4th and 5th terms are equal and numerically greatest; each = 489888.

Q3. In the expansion of $(1 + x)^{43}$, the coefficients of the $(2r + 1)$ th and the $(r + 2)$ th terms are equal. Find r .

Solution:

- Coefficient of the $(2r + 1)$ th term = ${}^{43}C_{2r}$.
- Coefficient of the $(r + 2)$ th term = ${}^{43}C_{r+1}$.
- Given ${}^{43}C_{2r} = {}^{43}C_{r+1}$. If ${}^nC_x = {}^nC_y$, then $x = y$ or $x + y = n$.
- Case 1: $2r = r + 1$, so $r = 1$. (This makes both terms the same 3rd term, a trivial case.)
- Case 2: $2r + (r + 1) = 43$, so $3r = 42$, so $r = 14$.

Ans. $r = 14$ (the case $r = 1$ is trivial).

Q4. Let A be the sum of the odd-position terms and B the sum of the even-position terms in the expansion of $(x + a)^n$. Prove that $A^2 - B^2 = (x^2 - a^2)^n$. Also find 4AB.

Solution:

1. Odd-position terms are $T_1, T_3, T_5, \dots$ and even-position terms are $T_2, T_4, \dots$. So $(x + a)^n = A + B \dots$ (i)
2. In $(x - a)^n$, the sign of a changes. The odd-position terms contain even powers of a, so they do not change sign. The even-position terms contain odd powers of a, so they change sign.
3. Therefore $(x - a)^n = A - B \dots$ (ii)
4. Multiply (i) and (ii): $(A + B)(A - B) = (x + a)^n(x - a)^n$.
5. $A^2 - B^2 = [(x + a)(x - a)]^n = (x^2 - a^2)^n$.
6. For 4AB: $(A + B)^2 - (A - B)^2 = 4AB$, so $4AB = (x + a)^{2n} - (x - a)^{2n}$.
7. Check with $n = 2$: $A = x^2 + a^2$, $B = 2ax$. $A^2 - B^2 = x^4 - 2x^2a^2 + a^4 = (x^2 - a^2)^2 \checkmark$

Ans. $A^2 - B^2 = (x^2 - a^2)^n$ and $4AB = (x + a)^{2n} - (x - a)^{2n}$.

Part 2: Properties, Proofs and Applications

Q5. Prove that $C_0 - C_1 + C_2 - C_3 + \dots + (-1)^r C_r = (-1)^r \cdot {}^{n-1}C_r$.

Solution:

1. Use Pascal's rule: ${}^nC_k = {}^{n-1}C_k + {}^{n-1}C_{k-1}$.
2. So $(-1)^k C_k = (-1)^k {}^{n-1}C_k + (-1)^k {}^{n-1}C_{k-1}$.
3. Write this for $k = 0, 1, 2, \dots, r$ and add. Let $D_k = {}^{n-1}C_k$.
4. $k = 0$: D_0
5. $k = 1$: $-D_1 - D_0$
6. $k = 2$: $+D_2 + D_1$
7. ...
8. $k = r$: $(-1)^r D_r + (-1)^r D_{r-1}$
9. Every D from D_0 to D_{r-1} cancels with the next line. Only $(-1)^r D_r$ survives.
10. Check with $n = 4, r = 2$: $1 - 4 + 6 = 3 = {}^3C_2 \checkmark$

Ans. Left side = $(-1)^r \cdot {}^{n-1}C_r$. Hence proved.

Q6. Prove that $C_0 + 2C_1 + 3C_2 + \dots + (n + 1)C_n = (n + 2) \cdot 2^{n-1}$.

Solution:

1. The general term of the left side is $(r + 1) C_r$, for $r = 0$ to n .
2. Split it: $(r + 1) C_r = r C_r + C_r$.
3. So the sum = $\sum r C_r + \sum C_r$.
4. Since $r \cdot {}^nC_r = n \cdot {}^{n-1}C_{r-1}$, $\sum r C_r = n [{}^{n-1}C_0 + {}^{n-1}C_1 + \dots + {}^{n-1}C_{n-1}] = n \cdot 2^{n-1}$.
5. Also $\sum C_r = 2^n$.

6. $\text{Sum} = n \cdot 2^{n-1} + 2^n = 2^{n-1}(n + 2).$

7. Check with $n = 2$: $1 + 4 + 3 = 8 = 4 \times 2 \checkmark$

Ans. Left side $= (n + 2) \cdot 2^{n-1}$. Hence proved.

Q7. Show that $3^{2n+2} - 8n - 9$ is divisible by 64 for every positive integer n .

Solution:

1. Write $3^{2n+2} = 9^{n+1} = (1 + 8)^{n+1}$.
2. Expand: $(1 + 8)^{n+1} = 1 + {}^{n+1}C_1(8) + {}^{n+1}C_2(8)^2 + {}^{n+1}C_3(8)^3 + \dots$
3. $= 1 + 8(n + 1) + 64[{}^{n+1}C_2 + 8 \cdot {}^{n+1}C_3 + 8^2 \cdot {}^{n+1}C_4 + \dots]$
4. Simplify the first two terms: $1 + 8n + 8 = 8n + 9$.
5. So $3^{2n+2} - 8n - 9 = 64[{}^{n+1}C_2 + 8 \cdot {}^{n+1}C_3 + 8^2 \cdot {}^{n+1}C_4 + \dots]$.
6. The bracket is a whole number, so the result is a multiple of 64.

Ans. $3^{2n+2} - 8n - 9$ is divisible by 64. Hence proved.

Q8. Which is larger: $(1.1)^{10000}$ or 1000?

Solution:

1. Write $(1.1)^{10000} = (1 + 0.1)^{10000}$.
2. Expand: $(1 + 0.1)^{10000} = 1 + 10000(0.1) + {}^{10000}C_2(0.1)^2 + {}^{10000}C_3(0.1)^3 + \dots$
3. Every term in the expansion is positive, so the whole sum is greater than the sum of its first two terms.
4. First two terms: $1 + 10000 \times 0.1 = 1 + 1000 = 1001$.
5. Therefore $(1.1)^{10000} > 1001 > 1000$.

Ans. $(1.1)^{10000}$ is larger than 1000.

Q9. Find the remainder when 5^{99} is divided by 13, using the binomial theorem.

Solution:

1. Write $5^{99} = 5 \times 5^{98} = 5 \times (5^2)^{49} = 5 \times (25)^{49} = 5 \times (26 - 1)^{49}$.
2. Expand: $(26 - 1)^{49} = 26^{49} - {}^{49}C_1 26^{48} + \dots + {}^{49}C_{48}(26)(-1)^{48} + (-1)^{49}$.
3. Every term except the last has the factor 26, so it is a multiple of 13. Hence $(26 - 1)^{49} = 13m - 1$.
4. $5^{99} = 5(13m - 1) = 65m - 5$.
5. Write $-5 = -13 + 8$: $5^{99} = 65m - 13 + 8 = 13(5m - 1) + 8$.
6. Check: $5^4 = 625 = 13 \times 48 + 1$, so $5^{99} = (5^4)^{24} \times 5^3$ leaves the same remainder as $5^3 = 125$, which is 8 $\checkmark$

Ans. Remainder = 8.

Part 3: Any Index, Series and Approximations

Q10. Prove that the coefficient of x^r in the expansion of $(1 - x)^{-n}$ (n a positive integer) is ${}^{n+r-1}C_r$. Hence write the first four terms of $(1 - x)^{-4}$.

Solution:

1. Write $(1 - x)^{-n} = [1 + (-x)]^{-n}$. The general term for any index is $[m(m-1)(m-2) \dots (m-r+1)/r!] y^r$, with $m = -n$ and $y = -x$.
2. Numerator: $(-n)(-n-1)(-n-2) \dots (-n-r+1)$ has r negative factors $= (-1)^r n(n+1)(n+2) \dots (n+r-1)$.
3. Also $y^r = (-x)^r = (-1)^r x^r$. The two factors $(-1)^r$ multiply to $+1$.
4. So $T_{r+1} = [n(n+1)(n+2) \dots (n+r-1)/r!] x^r$.
5. Multiply and divide by $(n-1)!$: $n(n+1) \dots (n+r-1) = (n+r-1)!/(n-1)!$.
6. $T_{r+1} = (n+r-1)! / [(n-1)! r!] x^r = {}^{n+r-1}C_r x^r$.
7. For $n = 4$: coefficients are ${}^4C_0 = 1$, ${}^4C_1 = 4$, ${}^5C_2 = 10$, ${}^6C_3 = 20$.

Ans. Coefficient of $x^r = {}^{n+r-1}C_r$; $(1 - x)^{-4} = 1 + 4x + 10x^2 + 20x^3 + \dots$

Q11. Expand $\sqrt[4]{4+x}$ in ascending powers of x up to x^3 , and state the range of validity. Hence find $\sqrt[4]{4.16}$ correct to 4 decimal places.

Solution:

1. Take 4 common: $\sqrt[4]{4+x} = (4)^{1/2}(1+x/4)^{1/2} = 2(1+y)^{1/2}$, where $y = x/4$.
2. Expand: $(1+y)^{1/2} = 1 + y/2 - y^2/8 + y^3/16 - \dots$
3. Put $y = x/4$: $1 + x/8 - x^2/128 + x^3/1024 - \dots$
4. Multiply by 2: $\sqrt[4]{4+x} = 2 + x/4 - x^2/64 + x^3/512 - \dots$
5. Validity: $|y| < 1$, so $|x/4| < 1$, i.e. $|x| < 4$.
6. For $\sqrt[4]{4.16}$, put $x = 0.16$: $x/4 = 0.04$; $x^2/64 = 0.0256/64 = 0.0004$; $x^3/512 = 0.004096/512 = 0.000008$.
7. $\sqrt[4]{4.16} \approx 2 + 0.04 - 0.0004 + 0.000008 = 2.039608$

Ans. $\sqrt[4]{4+x} = 2 + x/4 - x^2/64 + x^3/512 - \dots$, valid for $|x| < 4$; $\sqrt[4]{4.16} \approx 2.0396$.

Q12. Find the approximate value of $\sqrt[9]{99}$ correct to 4 decimal places using the binomial theorem.

Solution:

1. Write $99 = 100 - 1 = 100(1 - 0.01)$. So $\sqrt[9]{99} = 10(1 - 0.01)^{1/9}$.
2. Expand $(1 - x)^{1/9} = 1 + (1/9)(-x) + [(1/9)(-1/2)/2!](x)^2 + [(1/9)(-1/2)(-3/2)/3!](x)^3 + \dots$
3. $= 1 - x/9 - x^2/36 - x^3/216 - \dots$
4. Put $x = 0.01$: $x/9 = 0.001111$; $x^2/36 = 0.0000278$; $x^3/216 = 0.000000463$.
5. $(0.99)^{1/9} \approx 1 - 0.001111 - 0.0000278 - 0.000000463 = 0.998841156$.
6. Multiply by 10: $\sqrt[9]{99} \approx 9.9884$.

Ans. $\sqrt{99} \approx 9.9499$.

Q13. Expand $1/[(1-x)(1-2x)]$ in ascending powers of x up to x^3 , and find the coefficient of x^r .

Solution:

1. Use partial fractions: $1/[(1-x)(1-2x)] = A/(1-x) + B/(1-2x)$.
2. Then $1 = A(1-2x) + B(1-x)$. Put $x = 1$: $1 = -A$, so $A = -1$. Put $x = 1/2$: $1 = B/2$, so $B = 2$.
3. So the expression $= 2/(1-2x) - 1/(1-x) = 2(1-2x)^{-1} - (1-x)^{-1}$.
4. Use $(1-y)^{-1} = 1 + y + y^2 + y^3 + \dots$: $2[1 + 2x + 4x^2 + 8x^3 + \dots] - [1 + x + x^2 + x^3 + \dots]$.
5. Subtract like powers: $(2-1) + (4-1)x + (8-1)x^2 + (16-1)x^3 + \dots = 1 + 3x + 7x^2 + 15x^3 + \dots$
6. General coefficient: $2 \cdot 2^r - 1 = 2^{r+1} - 1$. Validity: $|2x| < 1$, i.e. $|x| < 1/2$.
7. Check: $(1-x)(1-2x) = 1 - 3x + 2x^2$, and $1/(1-3x+2x^2) = 1 + 3x + (9-2)x^2 + \dots = 1 + 3x + 7x^2 + \dots$ ✓

Ans. $1 + 3x + 7x^2 + 15x^3 + \dots$; coefficient of $x^r = 2^{r+1} - 1$ (valid for $|x| < 1/2$).

Q14. Using the approximation $(1+x)^n \approx 1+nx$ for small x , solve: (a) The side of a cube is measured 1% too large. Find the approximate percentage error in its volume. (b) The time period of a pendulum is $T = 2\pi\sqrt{l/g}$. If its length l is increased by 2%, find the approximate percentage change in T .

Solution:

1. **(a)** Let the true side $= s$, so the measured side $= 1.01s = s(1 + 0.01)$.
2. Volume with measured side $= [s(1 + 0.01)]^3 = s^3(1 + 0.01)^3 \approx s^3(1 + 3 \times 0.01) = s^3(1.03)$.
3. So the volume is about 3% more than the true volume.
4. **(b)** New length $l' = l(1 + 0.02)$. New period $T' = 2\pi\sqrt{l'/g} = 2\pi\sqrt{l/g} \times (1 + 0.02)^{1/2} = T(1.02)^{1/2}$.
5. Apply the approximation with $n = 1/2$: $(1.02)^{1/2} \approx 1 + (1/2)(0.02) = 1.01$.
6. So $T' \approx 1.01 T$.
7. Note: the exact values are $(1.01)^3 = 1.0303$ and $(1.02)^{1/2} = 1.00995$, very close to the approximate results.

Ans. (a) About 3% increase in volume. (b) About 1% increase in the time period.