

# MATHEMATICS 1

## Chapter: Binomial Theorem

*Broad (Long Answer) Questions with Complete Solutions: Set 3*

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### Part 1: Terms, Coefficients and Relations

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**Q1. Find the term independent of  $x$  in the expansion of  $(1 + x)^6 (1 + 1/x)^4$ .**

**Solution:**

1. Rewrite the second bracket:  $(1 + 1/x)^4 = ((x + 1)/x)^4 = (1 + x)^4 / x^4$ .
2. So the product =  $(1 + x)^6 (1 + x)^4 / x^4 = (1 + x)^{10} / x^4$ .
3. A term of the product is independent of  $x$  when the term taken from  $(1 + x)^{10}$  is  $x^4$ , because it then cancels with the  $x^4$  in the denominator.
4. Coefficient of  $x^4$  in  $(1 + x)^{10} = {}^{10}C_4 = (10 \times 9 \times 8 \times 7)/(4 \times 3 \times 2 \times 1) = 210$ .

**Ans.** Term independent of  $x = 210$ .

**Q2. The 3rd term in the expansion of  $(x + x^{\log_{10} x})^5$  is  $10^6$ . Find  $x$ .**

**Solution:**

1.  $T_3 = {}^5C_2 x^3 (x^{\log_{10} x})^2 = 10 x^{3 + 2 \log_{10} x}$ .
2. Given  $10 x^{3 + 2 \log_{10} x} = 10^6$ , so  $x^{3 + 2 \log_{10} x} = 10^5$ .
3. Take  $\log_{10}$  of both sides and let  $L = \log_{10} x$ :  $(3 + 2L) L = 5$ , so  $2L^2 + 3L - 5 = 0$ .
4. Factorise:  $(2L + 5)(L - 1) = 0$ , so  $L = 1$  or  $L = -5/2$ .
5.  $L = 1$  gives  $x = 10$ .  $L = -5/2$  gives  $x = 10^{-5/2}$ .
6. Check  $x = 10$ :  $T_3 = 10 \times 10^3 \times (10^1)^2 = 10^6 \checkmark$

**Ans.**  $x = 10$  or  $x = 10^{-5/2}$ .

**Q3. If the coefficients of  $x^{r-1}$ ,  $x^r$  and  $x^{r+1}$  in the expansion of  $(1 + x)^n$  are in A.P., prove that  $n^2 - n(4r + 1) + 4r^2 - 2 = 0$ .**

**Solution:**

1. The coefficients are  ${}^nC_{r-1}$ ,  ${}^nC_r$ ,  ${}^nC_{r+1}$ . In an A.P.,  $2 \cdot {}^nC_r = {}^nC_{r-1} + {}^nC_{r+1}$ .
2. Divide throughout by  ${}^nC_r$ :  $2 = {}^nC_{r-1}/{}^nC_r + {}^nC_{r+1}/{}^nC_r$ .
3. Use  ${}^nC_{r-1}/{}^nC_r = r/(n - r + 1)$  and  ${}^nC_{r+1}/{}^nC_r = (n - r)/(r + 1)$ .
4.  $2 = r/(n - r + 1) + (n - r)/(r + 1)$ .
5. Multiply by  $(n - r + 1)(r + 1)$ :  $2(n - r + 1)(r + 1) = r(r + 1) + (n - r)(n - r + 1)$ .
6. Left side:  $2(nr + n - r^2 + 1) = 2nr + 2n - 2r^2 + 2$ .
7. Right side:  $r^2 + r + (n - r)^2 + (n - r) = n^2 - 2nr + 2r^2 + n$ .

8. Equate and bring all terms to one side:  $0 = n^2 - 4nr + 4r^2 - n - 2$ .
9. Rearrange:  $n^2 - n(4r + 1) + 4r^2 - 2 = 0$ .

**Ans.** Hence proved.

#### **Q4. Find the integral part of $(\sqrt{3} + 1)^6$ .**

**Solution:**

1. Expand both:  $(\sqrt{3} + 1)^6 + (\sqrt{3} - 1)^6 = 2[(\sqrt{3})^6 + {}^6C_2(\sqrt{3})^4 + {}^6C_4(\sqrt{3})^2 + 1]$ .
2.  $(\sqrt{3})^6 = 27, (\sqrt{3})^4 = 9, (\sqrt{3})^2 = 3$ .
3.  $= 2[27 + 15 \times 9 + 15 \times 3 + 1] = 2[27 + 135 + 45 + 1] = 2 \times 208 = 416$ .
4. Now  $0 < \sqrt{3} - 1 < 1$  (it is about 0.732), so  $0 < (\sqrt{3} - 1)^6 < 1$ .
5. Therefore  $(\sqrt{3} + 1)^6 = 416 - (\sqrt{3} - 1)^6$ , which lies between 415 and 416.
6. Numerically,  $(\sqrt{3} - 1)^6 \approx 0.154$ , so  $(\sqrt{3} + 1)^6 \approx 415.85$ .

**Ans.** The integral part is 415.

#### **Q5. If $(7 + 4\sqrt{3})^n = I + F$ , where $I$ is a positive integer and $0 \leq F < 1$ , prove that $(I + F)(1 - F) = 1$ .**

**Solution:**

1. Let  $f = (7 - 4\sqrt{3})^n$ . Since  $(7 + 4\sqrt{3})(7 - 4\sqrt{3}) = 49 - 48 = 1$ , we have  $0 < 7 - 4\sqrt{3} < 1$  (it is about 0.072), so  $0 < f < 1$ .
2. By the binomial theorem,  $(7 + 4\sqrt{3})^n + (7 - 4\sqrt{3})^n = 2[\text{terms with even powers of } 4\sqrt{3}]$ , which is a whole number.
3. So  $I + F + f$  is an integer.
4. Since  $0 \leq F < 1$  and  $0 < f < 1$ , we get  $0 < F + f < 2$ . The only integer in this range is 1, so  $F + f = 1$ , i.e.  $f = 1 - F$ .
5. Now  $(I + F)(1 - F) = (I + F)f = (7 + 4\sqrt{3})^n (7 - 4\sqrt{3})^n = [(7 + 4\sqrt{3})(7 - 4\sqrt{3})]^n = 1^n = 1$ .

**Ans.**  $(I + F)(1 - F) = 1$ . Hence proved.

## **Part 2: Properties, Proofs and Applications**

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#### **Q6. Prove that $C_0 - C_1/2 + C_2/3 - \dots + (-1)^n C_n/(n + 1) = 1/(n + 1)$ .**

**Solution:**

1. The general term is  $(-1)^r {}^nC_r/(r + 1)$ , for  $r = 0$  to  $n$ .
2. Since  ${}^nC_r/(r + 1) = {}^{n+1}C_{r+1}/(n + 1)$ , the sum  $= [1/(n + 1)] \sum (-1)^r {}^{n+1}C_{r+1}$ .
3. Written out:  $[1/(n + 1)] \times [{}^{n+1}C_1 - {}^{n+1}C_2 + {}^{n+1}C_3 - \dots + (-1)^n {}^{n+1}C_{n+1}]$ .
4. Now use  $(1 - 1)^{n+1} = 0 = {}^{n+1}C_0 - {}^{n+1}C_1 + {}^{n+1}C_2 - \dots + (-1)^{n+1} {}^{n+1}C_{n+1}$ .
5. Moving  ${}^{n+1}C_0 = 1$  to the other side gives  ${}^{n+1}C_1 - {}^{n+1}C_2 + \dots + (-1)^n {}^{n+1}C_{n+1} = 1$ .

6. So the sum =  $1/(n+1) \times 1$ .

7. Check with  $n = 1$ :  $C_0 - C_1/2 = 1 - 1/2 = 1/2 = 1/(1+1) \checkmark$

**Ans.** Left side =  $1/(n+1)$ . Hence proved.

**Q7. Prove that  $C_1 + 4C_2 + 9C_3 + \dots + n^2C_n = n(n+1)2^{n-2}$ .**

**Solution:**

1. The general term is  $r^2 \cdot {}^nC_r$ . Write  $r^2 = r(r-1) + r$ .
2. Show  $r(r-1) \cdot {}^nC_r = n(n-1) \cdot {}^{n-2}C_{r-2}$ :  $r(r-1) n!/[r!(n-r)!] = n!/[(r-2)!(n-r)!] = n(n-1)(n-2)!/[(r-2)!((n-2)-(r-2))!]$ .
3. So  $\sum r(r-1) {}^nC_r = n(n-1)[{}^{n-2}C_0 + {}^{n-2}C_1 + \dots + {}^{n-2}C_{n-2}] = n(n-1)2^{n-2}$ .
4. Also  $\sum r \cdot {}^nC_r = n \cdot 2^{n-1} = 2n \cdot 2^{n-2}$ .
5. Add:  $n(n-1)2^{n-2} + 2n \cdot 2^{n-2} = n2^{n-2}(n-1+2) = n(n+1)2^{n-2}$ .
6. Check with  $n = 3$ :  $1(3) + 4(3) + 9(1) = 24$  and  $3 \times 4 \times 2 = 24 \checkmark$

**Ans.** Left side =  $n(n+1)2^{n-2}$ . Hence proved.

**Q8. Prove that the coefficient of  $x^m$  in  $(1+x)^p + (1+x)^{p+1} + \dots + (1+x)^n$  is  ${}^{n+1}C_{m+1} - {}^pC_{m+1}$ . Hence find the coefficient of  $x^7$  in  $(1+x)^7 + (1+x)^8 + \dots + (1+x)^{12}$ .**

**Solution:**

1. The given sum is a G.P. with first term  $(1+x)^p$ , common ratio  $(1+x)$  and  $(n-p+1)$  terms.
2.  $S = (1+x)^p [(1+x)^{n-p+1} - 1] / [(1+x) - 1] = [(1+x)^{n+1} - (1+x)^p] / x$ .
3. The coefficient of  $x^m$  in  $S$  equals the coefficient of  $x^{m+1}$  in the numerator  $(1+x)^{n+1} - (1+x)^p$ .
4. That is  ${}^{n+1}C_{m+1} - {}^pC_{m+1}$ .
5. For the given problem,  $p = 7, n = 12, m = 7$ : coefficient =  ${}^{13}C_8 - {}^7C_8$ .
6.  ${}^7C_8 = 0$  (since  $8 > 7$ ), and  ${}^{13}C_8 = {}^{13}C_5 = (13 \times 12 \times 11 \times 10 \times 9)/120 = 1287$ .

**Ans.** Coefficient of  $x^m = {}^{n+1}C_{m+1} - {}^pC_{m+1}$ ; the required coefficient = 1287.

**Q9. Show that  $11^{n+2} + 12^{2n+1}$  is divisible by 133 for every positive integer  $n$ .**

**Solution:**

1. Write  $12^{2n+1} = 12 \times (12^2)^n = 12 \times 144^n = 12(133 + 11)^n$ .
2. Expand:  $(133 + 11)^n = 11^n + {}^nC_1(133)(11)^{n-1} + {}^nC_2(133)^2(11)^{n-2} + \dots = 11^n + 133k$ , where  $k$  is a whole number.
3. So  $12^{2n+1} = 12 \times 11^n + 133 \times 12k$ .
4. Also  $11^{n+2} = 121 \times 11^n$ .
5. Add:  $11^{n+2} + 12^{2n+1} = (121 + 12) \times 11^n + 133 \times 12k = 133 \times 11^n + 133 \times 12k$ .
6. Take 133 common:  $= 133(11^n + 12k)$ .

**Ans.** The expression is divisible by 133. Hence proved.

**Q10. Find the last two digits of  $3^{400}$ .**

**Solution:**

1. Write  $3^{400} = (3^4)^{100} = 81^{100} = (1 + 80)^{100}$ .
2. Expand:  $(1 + 80)^{100} = 1 + {}^{100}C_1(80) + {}^{100}C_2(80)^2 + {}^{100}C_3(80)^3 + \dots$
3.  ${}^{100}C_1 \times 80 = 8000$ , which is a multiple of 100.
4.  $80^2 = 6400$  is a multiple of 100, so every term from the third onwards is also a multiple of 100.
5. Hence  $3^{400} = 1 + 100m$  for some whole number  $m$ .

**Ans.** The last two digits are 01.

### Part 3: Any Index, Series and Applications

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**Q11. Expand  $(8 + 3x)^{1/3}$  in ascending powers of  $x$  up to  $x^3$ , and state its validity. Hence find the cube root of 8.24 correct to 4 decimal places.**

**Solution:**

1. Take 8 common:  $(8 + 3x)^{1/3} = 8^{1/3} (1 + 3x/8)^{1/3} = 2(1 + y)^{1/3}$ , where  $y = 3x/8$ .
2. Expand  $(1 + y)^{1/3} = 1 + y/3 + [(1/3)(-2/3)/2!]y^2 + [(1/3)(-2/3)(-5/3)/3!]y^3 = 1 + y/3 - y^2/9 + 5y^3/81 - \dots$
3. Substitute  $y = 3x/8$ :  $y/3 = x/8$ ;  $y^2/9 = x^2/64$ ;  $5y^3/81 = 5x^3/1536$ .
4. Multiply by 2:  $(8 + 3x)^{1/3} = 2 + x/4 - x^2/32 + 5x^3/768 - \dots$
5. Validity:  $|3x/8| < 1$ , i.e.  $|x| < 8/3$ .
6. For 8.24,  $8 + 3x = 8.24$  gives  $x = 0.08$ .
7.  $x/4 = 0.02$ ;  $x^2/32 = 0.0064/32 = 0.0002$ ;  $5x^3/768 = 5(0.000512)/768 \approx 0.0000033$ .
8.  $(8.24)^{1/3} \approx 2 + 0.02 - 0.0002 + 0.0000033 = 2.0198033$ .

**Ans.**  $(8 + 3x)^{1/3} = 2 + x/4 - x^2/32 + 5x^3/768 - \dots$ , valid for  $|x| < 8/3$ ;  $(8.24)^{1/3} \approx 2.0198$ .

**Q12. Expand  $(2 + x)/(1 - x)^3$  up to  $x^3$  and find the coefficient of  $x^r$ .**

**Solution:**

1. Write the expression as  $(2 + x)(1 - x)^{-3}$ , valid for  $|x| < 1$ .
2. Standard series:  $(1 - x)^{-3} = 1 + 3x + 6x^2 + 10x^3 + \dots + [(r + 1)(r + 2)/2] x^r + \dots$
3. Multiply by 2:  $2 + 6x + 12x^2 + 20x^3 + \dots$
4. Multiply by  $x$ :  $x + 3x^2 + 6x^3 + \dots$
5. Add like powers:  $2 + 7x + 15x^2 + 26x^3 + \dots$
6. Coefficient of  $x^r$ : from  $2(1 - x)^{-3}$  we get  $(r + 1)(r + 2)$ ; from  $x(1 - x)^{-3}$  we get  $r(r + 1)/2$ .
7. Total =  $(r + 1)(r + 2) + r(r + 1)/2 = (r + 1)[2(r + 2) + r]/2 = (r + 1)(3r + 4)/2$ .
8. Check with  $r = 3$ :  $(4)(13)/2 = 26 \checkmark$

**Ans.**  $2 + 7x + 15x^2 + 26x^3 + \dots$ ; coefficient of  $x^r = (r + 1)(3r + 4)/2$ .

**Q13. Find the sum of the infinite series: (i)  $1 + \frac{2}{3} + \frac{3}{9} + \frac{4}{27} + \dots$  (ii)  $1 + \frac{3}{4} + \frac{6}{16} + \frac{10}{64} + \dots$**

**Solution:**

1. **(i)** Recall  $(1 - x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots$
2. Compare with  $1 + 2(\frac{1}{3}) + 3(\frac{1}{3})^2 + 4(\frac{1}{3})^3 + \dots$ : here  $x = \frac{1}{3}$ , which satisfies  $|x| < 1$ .
3. Sum  $= (1 - \frac{1}{3})^{-2} = (\frac{2}{3})^{-2} = \frac{9}{4}$ .
4. **(ii)** Recall  $(1 - x)^{-3} = 1 + 3x + 6x^2 + 10x^3 + \dots$
5. Compare with  $1 + 3(\frac{1}{4}) + 6(\frac{1}{4})^2 + 10(\frac{1}{4})^3 + \dots$ : here  $x = \frac{1}{4}$ .
6. Sum  $= (1 - \frac{1}{4})^{-3} = (\frac{3}{4})^{-3} = \frac{64}{27}$ .

**Ans.** (i)  $\frac{9}{4}$  (ii)  $\frac{64}{27}$ .

**Q14. Use the binomial approximation  $(1 + x)^n \approx 1 + nx$  ( $x$  small) to solve: (a) The radius of a sphere is increased by 0.5%. Find the approximate percentage increase in its volume and in its surface area. (b) The resistance of a wire is  $R = k/d^2$ , where  $d$  is its diameter. If  $d$  is decreased by 2%, find the approximate percentage change in  $R$ .**

**Solution:**

1. **(a)** New radius  $r' = r(1 + 0.005)$ .
2. Volume  $V = \frac{4}{3}\pi r^3$ , so  $V' = V(1 + 0.005)^3 \approx V(1 + 3 \times 0.005) = V(1.015)$ . Volume increases by about 1.5%.
3. Surface area  $S = 4\pi r^2$ , so  $S' = S(1 + 0.005)^2 \approx S(1 + 2 \times 0.005) = S(1.01)$ . Surface area increases by about 1%.
4. **(b)** New diameter  $d' = d(1 - 0.02)$ . New resistance  $R' = k/(d')^2 = (k/d^2)(1 - 0.02)^{-2} = R(1 - 0.02)^{-2}$ .
5. Apply the approximation with  $n = -2$ :  $(1 - 0.02)^{-2} \approx 1 + (-2)(-0.02) = 1.04$ .
6. So  $R' \approx 1.04 R$ .
7. Note: the exact value of  $(0.98)^{-2}$  is 1.0412, so the approximation is very close.

**Ans.** (a) Volume increases by about 1.5% and surface area by about 1%. (b) Resistance increases by about 4%.