

MATHEMATICS 1

Chapter: Complex Numbers

Long Answer Questions with Solutions | All Types of Broad Questions

This document contains only broad (long answer) questions of the chapter Complex Numbers. The questions are grouped by type, so you can practise every kind of long question that can come in the exam. Each question is worth 6 marks and is solved step by step. Read the short method note at the start of each type before attempting the questions.

Types covered:

- Type 1: Theory and proofs of fundamental results
- Type 2: Algebra of complex numbers (real and imaginary parts, modulus)
- Type 3: Polar and exponential form
- Type 4: De Moivre's theorem – simplification and proofs
- Type 5: Roots of complex numbers and equations
- Type 6: Cube roots of unity
- Type 7: Expansion of $\cos n\theta$, $\sin n\theta$, $\tan n\theta$
- Type 8: Expansion of powers of $\sin\theta$ and $\cos\theta$ in multiple angles
- Type 9: Locus in the Argand plane

Type 1: Theory and Proofs of Fundamental Results

Method: write the given data, state the formula, and prove step by step. Theory questions carry full marks only when every step is shown.

Q1. State and prove De Moivre's theorem for a positive integer n. [6 Marks]

Solution:

1. Statement: for every positive integer n, $(\cos\theta + i \sin\theta)^n = \cos n\theta + i \sin n\theta$.
2. Step 1 ($n = 1$): LHS = $\cos\theta + i \sin\theta$ = RHS. So the result is true for $n = 1$.
3. Step 2: Assume the result is true for $n = k$, i.e. $(\cos\theta + i \sin\theta)^k = \cos k\theta + i \sin k\theta$.
4. Step 3 ($n = k + 1$): $(\cos\theta + i \sin\theta)^{k+1} = (\cos\theta + i \sin\theta)^k(\cos\theta + i \sin\theta) = (\cos k\theta + i \sin k\theta)(\cos\theta + i \sin\theta)$.
5. Multiply out: $= (\cos k\theta \cos\theta - \sin k\theta \sin\theta) + i(\sin k\theta \cos\theta + \cos k\theta \sin\theta)$.
6. Use $\cos(A + B)$ and $\sin(A + B)$ formulae: $= \cos(k + 1)\theta + i \sin(k + 1)\theta$.
7. So the result is true for $n = k + 1$. By the principle of mathematical induction it is true for all positive integers n.

Answer: Hence proved.

Note: For a negative integer $n = -m$: $(\cos\theta + i \sin\theta)^{-m} = 1/(\cos m\theta + i \sin m\theta) = \cos m\theta - i \sin m\theta = \cos(-m\theta) + i \sin(-m\theta)$, so the theorem holds for all integers.

Q2. If $z_1 = r_1(\cos\theta_1 + i \sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i \sin\theta_2)$, prove that $|z_1 z_2| = |z_1||z_2|$, $\arg(z_1 z_2) = \arg z_1 + \arg z_2$, $|z_1/z_2| = |z_1|/|z_2|$ and $\arg(z_1/z_2) = \arg z_1 - \arg z_2$. [6 Marks]

Solution:

1. Product: $z_1 z_2 = r_1 r_2 (\cos\theta_1 + i \sin\theta_1)(\cos\theta_2 + i \sin\theta_2)$.
2. Multiply out: $= r_1 r_2 [(\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2) + i(\sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2)] = r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)]$.
3. This is in polar form, so $|z_1 z_2| = r_1 r_2 = |z_1||z_2|$ and $\arg(z_1 z_2) = \theta_1 + \theta_2$.
1. Quotient: $z_1/z_2 = (r_1/r_2) \times (\cos\theta_1 + i \sin\theta_1)(\cos\theta_2 - i \sin\theta_2) / [(\cos\theta_2 + i \sin\theta_2)(\cos\theta_2 - i \sin\theta_2)]$.
2. Denominator $= \cos^2\theta_2 + \sin^2\theta_2 = 1$. Numerator $= (\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2) + i(\sin\theta_1 \cos\theta_2 - \cos\theta_1 \sin\theta_2) = \cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)$.
3. So $z_1/z_2 = (r_1/r_2)[\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)]$, giving $|z_1/z_2| = r_1/r_2$ and $\arg(z_1/z_2) = \theta_1 - \theta_2$.

Answer: Hence proved.

Note: Geometrically, multiplying by z_2 scales the modulus by r_2 and rotates the point through the angle θ_2 .

Q3. Prove that the sum of all the n th roots of unity is zero ($n > 1$). Verify it for $n = 4$. [6 Marks]

Solution:

1. Let $x^n = 1 = \cos(2k\pi) + i \sin(2k\pi)$. By De Moivre, $x = \cos(2k\pi/n) + i \sin(2k\pi/n)$, $k = 0, 1, 2, \dots, n-1$.
2. Let $\alpha = \cos(2\pi/n) + i \sin(2\pi/n)$. Then by De Moivre $\alpha^k = \cos(2k\pi/n) + i \sin(2k\pi/n)$, so the n roots are $1, \alpha, \alpha^2, \dots, \alpha^{n-1}$.
3. These are in G.P. with first term 1 and common ratio α ($\alpha \neq 1$ as $n > 1$). Sum $= (1 - \alpha^n)/(1 - \alpha)$.
4. $\alpha^n = \cos 2\pi + i \sin 2\pi = 1$, so the numerator $1 - \alpha^n = 0$. Hence the sum $= 0$.
5. Verification for $n = 4$: roots are $\cos(k\pi/2) + i \sin(k\pi/2) = 1, i, -1, -i$, and $1 + i - 1 - i = 0$.

Answer: Sum of the n th roots of unity = 0 (proved).

Note: For $n = 3$ this gives the familiar result $1 + \omega + \omega^2 = 0$.

Type 2: Algebra of Complex Numbers

Method: put $z = x + iy$, rationalise using the conjugate, and equate real and imaginary parts. For questions on modulus use $|z_1/z_2| = |z_1|/|z_2|$ to save time.

Q4. If $(x + iy)^{1/3} = a + ib$, prove that $x/a + y/b = 4(a^2 - b^2)$. [6 Marks]

Solution:

1. Cube both sides: $x + iy = (a + ib)^3$.
2. Expand: $(a + ib)^3 = a^3 + 3a^2(ib) + 3a(ib)^2 + (ib)^3 = a^3 + 3ia^2b - 3ab^2 - ib^3$.
3. Separate parts: $(a^3 - 3ab^2) + i(3a^2b - b^3)$.

4. Equate real and imaginary parts: $x = a^3 - 3ab^2 = a(a^2 - 3b^2)$ and $y = 3a^2b - b^3 = b(3a^2 - b^2)$.
5. Divide: $x/a = a^2 - 3b^2$ and $y/b = 3a^2 - b^2$.
6. Add: $x/a + y/b = 4a^2 - 4b^2 = 4(a^2 - b^2)$.

Answer: Hence proved.

Q5. If $z = x + iy$ and $(z - 1)/(z + 1)$ is purely imaginary, prove that $|z| = 1$. [6 Marks]

Solution:

1. $(z - 1)/(z + 1) = [(x - 1) + iy]/[(x + 1) + iy]$. Multiply numerator and denominator by the conjugate $(x + 1) - iy$.
2. Numerator: $[(x - 1) + iy][(x + 1) - iy] = (x - 1)(x + 1) - iy(x - 1) + iy(x + 1) - i^2y^2 = (x^2 + y^2 - 1) + 2iy$.
3. Denominator: $(x + 1)^2 + y^2$.
4. So $(z - 1)/(z + 1) = (x^2 + y^2 - 1)/[(x + 1)^2 + y^2] + i \times 2y/[(x + 1)^2 + y^2]$.
5. The number is purely imaginary, so its real part is zero: $x^2 + y^2 - 1 = 0$, i.e. $x^2 + y^2 = 1$.
6. Therefore $|z| = \sqrt{x^2 + y^2} = 1$.

Answer: $|z| = 1$ (proved).

Note: If the expression were purely real instead, the imaginary part $2y$ would be zero, i.e. z would be real.

Q6. If $a + ib = (x + i)^2/(2x + 1 + i)$, prove that $a^2 + b^2 = (x^2 + 1)^2/(4x^2 + 4x + 2)$. [6 Marks]

Solution:

1. Take the modulus of both sides and use $|z_1/z_2| = |z_1|/|z_2|$: $|a + ib| = |x + i|^2/|2x + 1 + i|$.
2. $|a + ib| = \sqrt{a^2 + b^2}$.
3. $|x + i|^2 = (\sqrt{x^2 + 1})^2 = x^2 + 1$.
4. $|2x + 1 + i| = \sqrt{(2x + 1)^2 + 1} = \sqrt{4x^2 + 4x + 2}$.
5. So $\sqrt{a^2 + b^2} = (x^2 + 1)/\sqrt{4x^2 + 4x + 2}$.
6. Square both sides: $a^2 + b^2 = (x^2 + 1)^2/(4x^2 + 4x + 2)$.

Answer: Hence proved.

Type 3: Polar and Exponential Form

Method: find $r = |z|$ and $\theta = \arg z$ (check the quadrant). For products, quotients and powers, multiply or divide the moduli and add or subtract the arguments.

Q7. Find the modulus and argument of $z = (1 + i\sqrt{3})/(1 - i)$ using polar form. Also verify the result by rationalising. [6 Marks]

Solution:

1. Numerator $1 + i\sqrt{3}$: $r_1 = \sqrt{1 + 3} = 2$, $\theta_1 = \tan^{-1}\sqrt{3} = \pi/3$ (first quadrant).
2. Denominator $1 - i$: $r_2 = \sqrt{2}$, $\theta_2 = -\pi/4$ (fourth quadrant).

3. $|z| = r_1/r_2 = 2/\sqrt{2} = \sqrt{2}$.
4. $\arg z = \theta_1 - \theta_2 = \pi/3 + \pi/4 = 7\pi/12$.
5. Polar form: $z = \sqrt{2}[\cos(7\pi/12) + i \sin(7\pi/12)]$.
6. Verification: $z = (1 + i\sqrt{3})(1 + i)/[(1 - i)(1 + i)] = (1 + i + i\sqrt{3} - \sqrt{3})/2 = [(1 - \sqrt{3}) + i(1 + \sqrt{3})]/2$.
7. Modulus $= \sqrt{\{(1 - \sqrt{3})^2 + (1 + \sqrt{3})^2\}/4} = \sqrt{\{4 - 2\sqrt{3} + 4 + 2\sqrt{3}\}/4} = \sqrt{2}$. The real part is negative and the imaginary part is positive, so z is in the second quadrant, which is consistent with $7\pi/12 = 105^\circ$.

Answer: $|z| = \sqrt{2}$, $\arg z = 7\pi/12$

Q8. If $z_1 = 2e^{i\pi/3}$ and $z_2 = 3e^{i\pi/6}$, find z_1z_2 , z_1/z_2 and z_1^3 in the form $a + ib$. [6 Marks]

Solution:

1. Use the rule: $r_1e^{i\theta_1} \times r_2e^{i\theta_2} = r_1r_2e^{i(\theta_1 + \theta_2)}$.
2. $z_1z_2 = (2)(3)e^{i(\pi/3 + \pi/6)} = 6e^{i\pi/2} = 6(\cos\pi/2 + i \sin\pi/2) = 6i$.
3. $z_1/z_2 = (2/3)e^{i(\pi/3 - \pi/6)} = (2/3)e^{i\pi/6} = (2/3)(\cos\pi/6 + i \sin\pi/6) = (2/3)(\sqrt{3}/2 + i/2) = \sqrt{3}/3 + i/3$.
4. $z_1^3 = 2^3e^{i(3\pi/3)} = 8e^{i\pi} = 8(\cos\pi + i \sin\pi) = -8$.

Answer: $z_1z_2 = 6i$, $z_1/z_2 = 1/\sqrt{3} + i/3$, $z_1^3 = -8$

Type 4: De Moivre's Theorem – Simplification and Proofs

Method: convert every factor to the form $\cos n\theta + i \sin n\theta$ (remember $\cos n\theta - i \sin n\theta = \cos(-n\theta) + i \sin(-n\theta)$), then apply the index laws. For proofs, take $x = \cos\alpha + i \sin\alpha$ so that $x^n + 1/x^n = 2\cos n\alpha$.

Q9. Simplify $[(\cos 5\theta - i \sin 5\theta)^2 (\cos 3\theta + i \sin 3\theta)^{-5}] / [(\cos 4\theta + i \sin 4\theta)^3 (\cos 6\theta - i \sin 6\theta)^{-2}]$. [6 Marks]

Solution:

1. Write every factor as $\text{cis}(\text{angle})$, where $\text{cis}\phi = \cos\phi + i \sin\phi$. Note $\cos n\theta - i \sin n\theta = \text{cis}(-n\theta)$.
2. $(\cos 5\theta - i \sin 5\theta)^2 = [\text{cis}(-5\theta)]^2 = \text{cis}(-10\theta)$.
3. $(\cos 3\theta + i \sin 3\theta)^{-5} = [\text{cis}3\theta]^{-5} = \text{cis}(-15\theta)$. So numerator $= \text{cis}(-10\theta - 15\theta) = \text{cis}(-25\theta)$.
4. $(\cos 4\theta + i \sin 4\theta)^3 = \text{cis}12\theta$.
5. $(\cos 6\theta - i \sin 6\theta)^{-2} = [\text{cis}(-6\theta)]^{-2} = \text{cis}12\theta$. So denominator $= \text{cis}(12\theta + 12\theta) = \text{cis}24\theta$.
6. Ratio $= \text{cis}(-25\theta - 24\theta) = \text{cis}(-49\theta) = \cos(-49\theta) + i \sin(-49\theta)$.

Answer: $\cos 49\theta - i \sin 49\theta$

Q10. Prove that $(1 + i\sqrt{3})^n + (1 - i\sqrt{3})^n = 2^{n+1} \cos(n\pi/3)$. Hence evaluate $(1 + i\sqrt{3})^8 + (1 - i\sqrt{3})^8$. [6 Marks]

Solution:

1. For $1 + i\sqrt{3}$: $r = 2$, $\theta = \pi/3$, so $1 + i\sqrt{3} = 2(\cos\pi/3 + i \sin\pi/3)$.
2. For $1 - i\sqrt{3}$: $r = 2$, $\theta = -\pi/3$, so $1 - i\sqrt{3} = 2(\cos\pi/3 - i \sin\pi/3)$.

3. By De Moivre: $(1 + i\sqrt{3})^n = 2^n(\cos n\pi/3 + i \sin n\pi/3)$ and $(1 - i\sqrt{3})^n = 2^n(\cos n\pi/3 - i \sin n\pi/3)$.
4. Add: $\text{sum} = 2^n \times 2\cos(n\pi/3) = 2^{n+1}\cos(n\pi/3)$.
5. Put $n = 8$: $\text{sum} = 2^9 \cos(8\pi/3)$. Now $8\pi/3 = 2\pi + 2\pi/3$, so $\cos(8\pi/3) = \cos(2\pi/3) = -1/2$.
6. $\text{Sum} = 512 \times (-1/2) = -256$.

Answer: Proved; the value for $n = 8$ is -256 .

Q11. If $x = \cos\alpha + i \sin\alpha$ and $y = \cos\beta + i \sin\beta$, prove that (i) $x^m y^n + 1/(x^m y^n) = 2\cos(m\alpha + n\beta)$ (ii) $(x - y)/(x + y) = i \tan[(\alpha - \beta)/2]$. [6 Marks]

Solution:

1. (i) By De Moivre: $x^m = \cos m\alpha + i \sin m\alpha$ and $y^n = \cos n\beta + i \sin n\beta$.
2. So $x^m y^n = \cos(m\alpha + n\beta) + i \sin(m\alpha + n\beta)$, and its reciprocal $1/(x^m y^n) = \cos(m\alpha + n\beta) - i \sin(m\alpha + n\beta)$.
3. Add: $x^m y^n + 1/(x^m y^n) = 2\cos(m\alpha + n\beta)$.
4. (ii) Use Euler's form: $x = e^{i\alpha}$, $y = e^{i\beta}$. Let $\gamma = (\alpha - \beta)/2$.
5. Take $e^{i(\alpha+\beta)/2}$ common: $x = e^{i(\alpha+\beta)/2} \cdot e^{i\gamma}$ and $y = e^{i(\alpha+\beta)/2} \cdot e^{-i\gamma}$.
6. So $(x - y)/(x + y) = (e^{i\gamma} - e^{-i\gamma})/(e^{i\gamma} + e^{-i\gamma})$.
7. Using $e^{i\gamma} - e^{-i\gamma} = 2i \sin\gamma$ and $e^{i\gamma} + e^{-i\gamma} = 2\cos\gamma$: $\text{ratio} = 2i \sin\gamma/(2\cos\gamma) = i \tan\gamma = i \tan[(\alpha - \beta)/2]$.

Answer: Hence proved.

Q12. If $\cos\alpha + \cos\beta + \cos\gamma = \sin\alpha + \sin\beta + \sin\gamma = 0$, prove that (i) $\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha + \beta + \gamma)$ and $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha + \beta + \gamma)$ (ii) $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = 0$ and $\sin 2\alpha + \sin 2\beta + \sin 2\gamma = 0$. [6 Marks]

Solution:

1. Let $a = \cos\alpha + i \sin\alpha$, $b = \cos\beta + i \sin\beta$, $c = \cos\gamma + i \sin\gamma$.
2. $a + b + c = (\cos\alpha + \cos\beta + \cos\gamma) + i(\sin\alpha + \sin\beta + \sin\gamma) = 0 + i0 = 0$.
3. (i) Since $a + b + c = 0$, the identity $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$ gives $a^3 + b^3 + c^3 = 3abc$.
4. By De Moivre, $\text{LHS} = (\cos 3\alpha + \cos 3\beta + \cos 3\gamma) + i(\sin 3\alpha + \sin 3\beta + \sin 3\gamma)$. $\text{RHS} = 3abc = 3[\cos(\alpha + \beta + \gamma) + i \sin(\alpha + \beta + \gamma)]$.
5. Equate real and imaginary parts to get both results of (i).
6. (ii) Since $|a| = |b| = |c| = 1$, we have $1/a = \cos\alpha - i \sin\alpha = \text{conjugate of } a$, and similarly for b and c . So $1/a + 1/b + 1/c = \text{conjugate of } (a + b + c) = 0$.
7. That is $(bc + ca + ab)/(abc) = 0$, so $ab + bc + ca = 0$.
8. Now $a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + bc + ca) = 0 - 0 = 0$.
9. By De Moivre, $(\cos 2\alpha + \cos 2\beta + \cos 2\gamma) + i(\sin 2\alpha + \sin 2\beta + \sin 2\gamma) = 0$, so both parts are zero.

Answer: Hence proved.

Type 5: Roots of Complex Numbers and Equations

Method: write the number in polar form as $r[\cos(\theta + 2k\pi) + i \sin(\theta + 2k\pi)]$, apply the power $1/n$, and give $k = 0, 1, \dots, n - 1$. To solve an equation like $x^n = a$, treat it as finding the n th roots of a .

Q13. Find all the values of $(8i)^{1/3}$ and express them in the form $a + ib$. [6 Marks]

Solution:

1. $8i = 0 + 8i$: $r = 8$, $\theta = \pi/2$. So $8i = 8[\cos(\pi/2 + 2k\pi) + i \sin(\pi/2 + 2k\pi)]$.
2. $(8i)^{1/3} = 8^{1/3}[\cos((\pi/2 + 2k\pi)/3) + i \sin((\pi/2 + 2k\pi)/3)] = 2[\cos(\pi/6 + 2k\pi/3) + i \sin(\pi/6 + 2k\pi/3)]$, $k = 0, 1, 2$.
3. $k = 0$: $2(\cos\pi/6 + i \sin\pi/6) = 2(\sqrt{3}/2 + i/2) = \sqrt{3} + i$.
4. $k = 1$: angle $= \pi/6 + 2\pi/3 = 5\pi/6$: $2(\cos 5\pi/6 + i \sin 5\pi/6) = 2(-\sqrt{3}/2 + i/2) = -\sqrt{3} + i$.
5. $k = 2$: angle $= \pi/6 + 4\pi/3 = 3\pi/2$: $2(\cos 3\pi/2 + i \sin 3\pi/2) = 2(0 - i) = -2i$.

Answer: $\sqrt{3} + i$, $-\sqrt{3} + i$, $-2i$

Note: Check: $(-2i)^3 = -8i^3 = -8(-i) = 8i$. Also the sum of the three roots is zero, as it should be.

Q14. Solve $x^4 + 1 = 0$ and hence factorise $x^4 + 1$ into two real quadratic factors. [6 Marks]

Solution:

1. $x^4 = -1 = \cos(\pi + 2k\pi) + i \sin(\pi + 2k\pi)$.
2. $x = \cos((2k + 1)\pi/4) + i \sin((2k + 1)\pi/4)$, $k = 0, 1, 2, 3$.
3. $k = 0$: $\cos\pi/4 + i \sin\pi/4 = (1 + i)/\sqrt{2}$. $k = 1$: $\cos 3\pi/4 + i \sin 3\pi/4 = (-1 + i)/\sqrt{2}$.
4. $k = 2$: $\cos 5\pi/4 + i \sin 5\pi/4 = (-1 - i)/\sqrt{2}$. $k = 3$: $\cos 7\pi/4 + i \sin 7\pi/4 = (1 - i)/\sqrt{2}$.
5. Factorisation: pair the conjugate roots. For $(1 \pm i)/\sqrt{2}$: sum $= \sqrt{2}$, product $= (1 + 1)/2 = 1$, giving the factor $x^2 - \sqrt{2}x + 1$.
6. For $(-1 \pm i)/\sqrt{2}$: sum $= -\sqrt{2}$, product $= 1$, giving the factor $x^2 + \sqrt{2}x + 1$.
7. Check: $(x^2 + 1)^2 - (\sqrt{2}x)^2 = x^4 + 2x^2 + 1 - 2x^2 = x^4 + 1$.

Answer: Roots: $(\pm 1 \pm i)/\sqrt{2}$ (all four sign combinations); $x^4 + 1 = (x^2 - \sqrt{2}x + 1)(x^2 + \sqrt{2}x + 1)$

Q15. Solve $z^6 + 64 = 0$ and write the roots in the form $a + ib$. [6 Marks]

Solution:

1. $z^6 = -64 = 64[\cos(\pi + 2k\pi) + i \sin(\pi + 2k\pi)]$.
2. $z = 64^{1/6}[\cos((2k + 1)\pi/6) + i \sin((2k + 1)\pi/6)] = 2[\cos((2k + 1)\pi/6) + i \sin((2k + 1)\pi/6)]$, $k = 0, 1, 2, 3, 4, 5$.
3. $k = 0$: $2\text{cis}(\pi/6) = \sqrt{3} + i$. $k = 1$: $2\text{cis}(\pi/2) = 2i$. $k = 2$: $2\text{cis}(5\pi/6) = -\sqrt{3} + i$.
4. $k = 3$: $2\text{cis}(7\pi/6) = -\sqrt{3} - i$. $k = 4$: $2\text{cis}(3\pi/2) = -2i$. $k = 5$: $2\text{cis}(11\pi/6) = \sqrt{3} - i$.

Answer: $z = \pm\sqrt{3} \pm i$ (all four sign combinations), $2i$, $-2i$

Note: All six roots lie on a circle of radius 2 and are 60° apart.

Type 6: Cube Roots of Unity

Method: use $\omega^3 = 1$ and $1 + \omega + \omega^2 = 0$. Reduce every power of ω by dividing the index by 3, and replace $1 + \omega$ by $-\omega^2$, or $1 + \omega^2$ by $-\omega$.

Q16. If ω is a complex cube root of unity, prove that (i) $(a + b\omega + c\omega^2)/(c + a\omega + b\omega^2) = \omega^2$ (ii) $(1 - \omega)(1 - \omega^2)(1 - \omega^4)(1 - \omega^5) = 9$. [6 Marks]

Solution:

1. Recall $\omega^3 = 1$, so $\omega^4 = \omega$ and $\omega^5 = \omega^2$.
2. (i) Multiply the denominator by ω^2 : $\omega^2(c + a\omega + b\omega^2) = c\omega^2 + a\omega^3 + b\omega^4 = c\omega^2 + a + b\omega = a + b\omega + c\omega^2$.
3. So numerator = $\omega^2 \times$ denominator, and the ratio is ω^2 .
4. (ii) Replace ω^4 by ω and ω^5 by ω^2 : product = $(1 - \omega)(1 - \omega^2)(1 - \omega)(1 - \omega^2) = [(1 - \omega)(1 - \omega^2)]^2$.
5. $(1 - \omega)(1 - \omega^2) = 1 - \omega^2 - \omega + \omega^3 = 2 - (\omega + \omega^2) = 2 - (-1) = 3$.
6. Therefore the product = $3^2 = 9$.

Answer: (i) ω^2 (ii) 9 (both proved)

Q17. If ω is a cube root of unity, prove that $(x + y + z)(x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z) = x^3 + y^3 + z^3 - 3xyz$. Hence factorise $a^3 + b^3 + c^3 - 3abc$. [6 Marks]

Solution:

1. First multiply the last two brackets term by term: $(x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z)$.
2. $= x^2 + \omega^2 xy + \omega xz + \omega xy + \omega^3 y^2 + \omega^2 yz + \omega^2 xz + \omega^4 yz + \omega^3 z^2$.
3. Use $\omega^3 = 1$ and $\omega^4 = \omega$: $= x^2 + y^2 + z^2 + xy(\omega^2 + \omega) + xz(\omega + \omega^2) + yz(\omega^2 + \omega)$.
4. Since $\omega + \omega^2 = -1$: $= x^2 + y^2 + z^2 - xy - yz - zx$.
5. Multiply by $(x + y + z)$: $(x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx) = x^3 + y^3 + z^3 - 3xyz$.
6. Hence, replacing x, y, z by a, b, c : $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a + \omega b + \omega^2 c)(a + \omega^2 b + \omega c)$.

Answer: Hence proved.

Type 7: Expansion of $\cos n\theta$, $\sin n\theta$ and $\tan n\theta$

Method: write $\cos n\theta + i \sin n\theta = (\cos\theta + i \sin\theta)^n$, expand by the binomial theorem, and equate real and imaginary parts. Then use $\sin^2\theta = 1 - \cos^2\theta$ (or $\cos^2\theta = 1 - \sin^2\theta$) to get a single function. For $\tan n\theta$, divide the numerator and denominator by $\cos^n\theta$.

Q18. Prove that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$ and $\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$. [6 Marks]

Solution:

1. Let $c = \cos\theta$, $s = \sin\theta$. $\cos 5\theta + i \sin 5\theta = (c + is)^5$.

2. Expand: $c^5 + 5c^4(is) + 10c^3(is)^2 + 10c^2(is)^3 + 5c(is)^4 + (is)^5 = c^5 + 5ic^4s - 10c^3s^2 - 10ic^2s^3 + 5cs^4 + is^5$.
3. Real part: $\cos 5\theta = c^5 - 10c^3s^2 + 5cs^4$. Imaginary part: $\sin 5\theta = 5c^4s - 10c^2s^3 + s^5$.
4. For $\cos 5\theta$ put $s^2 = 1 - c^2$: $c^5 - 10c^3(1 - c^2) + 5c(1 - c^2)^2 = c^5 - 10c^3 + 10c^5 + 5c - 10c^3 + 5c^5 = 16c^5 - 20c^3 + 5c$.
5. For $\sin 5\theta$ put $c^2 = 1 - s^2$: $5(1 - s^2)^2s - 10(1 - s^2)s^3 + s^5 = 5s - 10s^3 + 5s^5 - 10s^3 + 10s^5 + s^5 = 16s^5 - 20s^3 + 5s$.

Answer: $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$; $\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$ (proved)

Q19. Expand $\cos 6\theta$, $\sin 6\theta$ and $\tan 6\theta$ using De Moivre's theorem. [6 Marks]

Solution:

1. Let $c = \cos\theta$, $s = \sin\theta$. $\cos 6\theta + i \sin 6\theta = (c + is)^6$.
2. Binomial expansion: $c^6 + 6ic^5s + 15i^2c^4s^2 + 20i^3c^3s^3 + 15i^4c^2s^4 + 6i^5cs^5 + i^6s^6$.
3. Simplify with $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$, $i^6 = -1$: $(c^6 - 15c^4s^2 + 15c^2s^4 - s^6) + i(6c^5s - 20c^3s^3 + 6cs^5)$.
4. So $\cos 6\theta = c^6 - 15c^4s^2 + 15c^2s^4 - s^6$ and $\sin 6\theta = 6c^5s - 20c^3s^3 + 6cs^5$.
5. Put $s^2 = 1 - c^2$ in $\cos 6\theta$: $c^6 - 15c^4(1 - c^2) + 15c^2(1 - c^2)^2 - (1 - c^2)^3 = c^6 - 15c^4 + 15c^6 + 15c^2 - 30c^4 + 15c^6 - 1 + 3c^2 - 3c^4 + c^6 = 32c^6 - 48c^4 + 18c^2 - 1$.
6. $\tan 6\theta = \sin 6\theta / \cos 6\theta$. Divide numerator and denominator by c^6 and put $t = \tan\theta$: $\tan 6\theta = (6t - 20t^3 + 6t^5) / (1 - 15t^2 + 15t^4 - t^6)$.

Answer: $\cos 6\theta = 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$; $\sin 6\theta = 6\cos^5\theta \sin\theta - 20\cos^3\theta \sin^3\theta + 6\cos\theta \sin^5\theta$; $\tan 6\theta = (6\tan\theta - 20\tan^3\theta + 6\tan^5\theta) / (1 - 15\tan^2\theta + 15\tan^4\theta - \tan^6\theta)$

Type 8: Powers of $\sin\theta$ and $\cos\theta$ in Multiple Angles

Method: put $z = \cos\theta + i \sin\theta$. Then $2\cos\theta = z + 1/z$ and $2i \sin\theta = z - 1/z$, and $z^n + 1/z^n = 2\cos n\theta$, $z^n - 1/z^n = 2i \sin n\theta$. Expand by the binomial theorem and group the terms in pairs (first and last, second and second-last, and so on).

Q20. Prove that $\cos^5\theta = (1/16)(\cos 5\theta + 5\cos 3\theta + 10\cos\theta)$. [6 Marks]

Solution:

1. Let $z = \cos\theta + i \sin\theta$, so $2\cos\theta = z + 1/z$. Then $(2\cos\theta)^5 = (z + 1/z)^5$, and LHS = $32\cos^5\theta$.
2. Expand: $(z + 1/z)^5 = z^5 + 5z^3 + 10z + 10/z + 5/z^3 + 1/z^5$.
3. Group in pairs: $= (z^5 + 1/z^5) + 5(z^3 + 1/z^3) + 10(z + 1/z) = 2\cos 5\theta + 10\cos 3\theta + 20\cos\theta$.
4. So $32\cos^5\theta = 2\cos 5\theta + 10\cos 3\theta + 20\cos\theta$. Divide by 32.

Answer: $\cos^5\theta = (1/16)(\cos 5\theta + 5\cos 3\theta + 10\cos\theta)$

Note: Check with $\theta = 0$: LHS = 1 and RHS = $(1 + 5 + 10)/16 = 1$.

Q21. Express $\sin^6\theta$ in a series of cosines of multiples of θ . [6 Marks]

Solution:

1. Let $z = \cos\theta + i \sin\theta$, so $2i \sin\theta = z - 1/z$. Then $(2i \sin\theta)^6 = (z - 1/z)^6$.

2. $LHS = 64i^6 \sin^6 \theta = -64 \sin^6 \theta$ (since $i^6 = -1$).
3. Expand RHS: $z^6 - 6z^4 + 15z^2 - 20 + 15/z^2 - 6/z^4 + 1/z^6$.
4. Group in pairs: $= (z^6 + 1/z^6) - 6(z^4 + 1/z^4) + 15(z^2 + 1/z^2) - 20 = 2\cos 6\theta - 12\cos 4\theta + 30\cos 2\theta - 20$.
5. So $-64 \sin^6 \theta = 2\cos 6\theta - 12\cos 4\theta + 30\cos 2\theta - 20$. Divide by -64 : $\sin^6 \theta = (1/32)(10 - 15\cos 2\theta + 6\cos 4\theta - \cos 6\theta)$.

Answer: $\sin^6 \theta = (1/32)(10 - 15\cos 2\theta + 6\cos 4\theta - \cos 6\theta)$

Note: Check with $\theta = \pi/2$: $LHS = 1$ and $RHS = (10 + 15 + 6 + 1)/32 = 1$. Note that for an even power of $\sin \theta$ the answer contains cosines, and for an odd power it contains sines.

Q22. Express $\cos^3 \theta \sin^2 \theta$ in a series of cosines of multiples of θ . [6 Marks]

Solution:

1. Let $z = \cos \theta + i \sin \theta$. Then $(2\cos \theta)^3 (2i \sin \theta)^2 = (z + 1/z)^3 (z - 1/z)^2$.
2. $LHS = 8\cos^3 \theta \times (-4 \sin^2 \theta) = -32 \cos^3 \theta \sin^2 \theta$.
3. RHS: write $(z + 1/z)^3 (z - 1/z)^2 = (z + 1/z)[(z + 1/z)(z - 1/z)]^2 = (z + 1/z)(z^2 - 1/z^2)^2$.
4. $(z^2 - 1/z^2)^2 = z^4 - 2 + 1/z^4$. Multiply by $(z + 1/z)$: $z^5 - 2z + 1/z^3 + z^3 - 2/z + 1/z^5$.
5. Group in pairs: $(z^5 + 1/z^5) + (z^3 + 1/z^3) - 2(z + 1/z) = 2\cos 5\theta + 2\cos 3\theta - 4\cos \theta$.
6. So $-32 \cos^3 \theta \sin^2 \theta = 2\cos 5\theta + 2\cos 3\theta - 4\cos \theta$. Divide by -32 : $\cos^3 \theta \sin^2 \theta = (1/16)(2\cos \theta - \cos 3\theta - \cos 5\theta)$.

Answer: $\cos^3 \theta \sin^2 \theta = (1/16)(2\cos \theta - \cos 3\theta - \cos 5\theta)$

Note: Check with $\theta = \pi/3$: $LHS = (1/8)(3/4) = 3/32$ and $RHS = (1/16)(1 + 1 - 1/2) = 3/32$.

Type 9: Locus in the Argand Plane

Method: put $z = x + iy$, use $|z| = \sqrt{x^2 + y^2}$ and the rationalised form of the given expression, then simplify to get the equation of a line or a circle.

Q23. Find the locus of the point $z = x + iy$ if (i) $\arg[(z - 1)/(z + 1)] = \pi/2$ (ii) $|z - 3| = 2|z + 1|$. [6 Marks]

Solution:

1. (i) Rationalising as in Q5: $(z - 1)/(z + 1) = (x^2 + y^2 - 1)/D + i \times 2y/D$, where $D = (x + 1)^2 + y^2 > 0$.
2. An argument of $\pi/2$ means the number lies on the positive imaginary axis: real part = 0 and imaginary part > 0 .
3. Real part = 0: $x^2 + y^2 = 1$. Imaginary part > 0 : $y > 0$.
4. (ii) Square both sides: $|z - 3|^2 = 4|z + 1|^2$, i.e. $(x - 3)^2 + y^2 = 4[(x + 1)^2 + y^2]$.
5. Expand: $x^2 - 6x + 9 + y^2 = 4x^2 + 8x + 4 + 4y^2$.
6. Simplify: $3x^2 + 3y^2 + 14x - 5 = 0$, i.e. $x^2 + y^2 + (14/3)x - 5/3 = 0$.

7. Complete the square: $(x + 7/3)^2 + y^2 = 5/3 + 49/9 = 64/9$.

Answer: (i) The upper half ($y > 0$) of the circle $x^2 + y^2 = 1$. (ii) The circle with centre $(-7/3, 0)$ and radius $8/3$.

Note: In (i) the circle has diameter joining the points 1 and -1 , so the angle at z is a right angle (angle in a semicircle).

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