

MATHEMATICS 1

Chapter: Complex Numbers

Long Answer Questions with Solutions | Set 2 | All Types of Broad Questions

This is Set 2 of the broad (long answer) questions of the chapter Complex Numbers. It contains completely new questions, different from Set 1, and follows the same type-wise arrangement. Each question is worth 6 marks and is solved step by step. Read the method note at the start of every type before attempting the questions.

Types covered:

- Type 1: Theory and proofs of fundamental results
- Type 2: Algebra of complex numbers and quadratic equations
- Type 3: Polar form, modulus, argument and powers
- Type 4: De Moivre's theorem and Euler's form – proofs
- Type 5: Roots of complex numbers and equations
- Type 6: Cube roots of unity
- Type 7: Expansion of $\cos n\theta$ and $\tan n\theta$
- Type 8: Expansion of powers of $\sin\theta$ and $\cos\theta$ in multiple angles
- Type 9: Locus in the Argand plane

Type 1: Theory and Proofs of Fundamental Results

Method: use $|z|^2 = z\bar{z}$, the fact that $w + \bar{w} = 2\operatorname{Re}(w)$, and expand carefully. Write the given data first and state the result to be proved.

Q1. Prove the triangle inequality $|z_1 + z_2| \leq |z_1| + |z_2|$ and the parallelogram law $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$. [6 Marks]

Solution:

1. Use $|z|^2 = z\bar{z}$ and $\operatorname{conj}(z_1 + z_2) = \bar{z}_1 + \bar{z}_2$.
2. $|z_1 + z_2|^2 = (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) = z_1\bar{z}_1 + z_2\bar{z}_2 + z_1\bar{z}_2 + \bar{z}_1z_2 = |z_1|^2 + |z_2|^2 + 2\operatorname{Re}(z_1\bar{z}_2)$ (1)
3. Here $\bar{z}_1z_2$ is the conjugate of $z_1\bar{z}_2$, and $w + \bar{w} = 2\operatorname{Re}(w)$.
4. Since $\operatorname{Re}(w) \leq |w|$, we get $2\operatorname{Re}(z_1\bar{z}_2) \leq 2|z_1\bar{z}_2| = 2|z_1||z_2|$.
5. From (1): $|z_1 + z_2|^2 \leq |z_1|^2 + |z_2|^2 + 2|z_1||z_2| = (|z_1| + |z_2|)^2$. Taking positive square roots, $|z_1 + z_2| \leq |z_1| + |z_2|$.
6. Parallelogram law: replace z_2 by $-z_2$ in (1): $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2\operatorname{Re}(z_1\bar{z}_2)$ (2)
7. Add (1) and (2): $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$.

Answer: Both results proved.

Note: Geometrically: the length of one side of a triangle is at most the sum of the other two sides, and in a parallelogram the sum of the squares of the diagonals equals the sum of the squares of all four sides.

Q2. Prove that (i) $\text{conj}(z_1 z_2) = \bar{z}_1 \bar{z}_2$ (ii) $\text{conj}(z_1/z_2) = \bar{z}_1/\bar{z}_2$ (iii) $z\bar{z} = |z|^2$ (iv) z is real if and only if $z = \bar{z}$, and z is purely imaginary if and only if $z = -\bar{z}$. [6 Marks]

Solution:

1. Let $z_1 = a + ib$ and $z_2 = c + id$.
2. (i) $z_1 z_2 = (ac - bd) + i(ad + bc)$, so $\text{conj}(z_1 z_2) = (ac - bd) - i(ad + bc)$.
3. Also $\bar{z}_1 \bar{z}_2 = (a - ib)(c - id) = ac - iad - ibc + i^2 bd = (ac - bd) - i(ad + bc)$. Hence $\text{conj}(z_1 z_2) = \bar{z}_1 \bar{z}_2$.
1. (ii) Let $w = z_1/z_2$, so $z_1 = wz_2$. By part (i), $\bar{z}_1 = \bar{w} \bar{z}_2$, so $\bar{w} = \bar{z}_1/\bar{z}_2$, i.e. $\text{conj}(z_1/z_2) = \bar{z}_1/\bar{z}_2$.
2. (iii) For $z = a + ib$: $z\bar{z} = (a + ib)(a - ib) = a^2 - i^2 b^2 = a^2 + b^2 = |z|^2$.
3. (iv) $z = \bar{z} \Leftrightarrow a + ib = a - ib \Leftrightarrow 2ib = 0 \Leftrightarrow b = 0$, so z is real. Similarly $z = -\bar{z} \Leftrightarrow a + ib = -a - ib \Leftrightarrow a = 0$, so z is purely imaginary.

Answer: All four results proved.

Q3. Prove that the product of all the n th roots of unity is $(-1)^{n-1}$. Verify for $n = 3$ and $n = 4$. [6 Marks]

Solution:

1. The n th roots of unity are $1, \alpha, \alpha^2, \dots, \alpha^{n-1}$, where $\alpha = \cos(2\pi/n) + i \sin(2\pi/n)$.
2. Product $P = 1 \cdot \alpha \cdot \alpha^2 \dots \alpha^{n-1} = \alpha^{1+2+\dots+(n-1)} = \alpha^{n(n-1)/2}$.
3. By De Moivre: $\alpha^{n(n-1)/2} = \cos[(2\pi/n) \times n(n-1)/2] + i \sin[(2\pi/n) \times n(n-1)/2] = \cos(n-1)\pi + i \sin(n-1)\pi$.
4. Since $\sin(n-1)\pi = 0$ and $\cos(n-1)\pi = (-1)^{n-1}$, we get $P = (-1)^{n-1}$.
5. $n = 3$: $1 \cdot \omega \cdot \omega^2 = \omega^3 = 1 = (-1)^2$.
6. $n = 4$: roots are $1, i, -1, -i$; product $= 1 \cdot i \cdot (-1) \cdot (-i) = i \cdot i = -1 = (-1)^3$.

Answer: Product of the n th roots of unity $= (-1)^{n-1}$ (proved).

Type 2: Algebra of Complex Numbers and Quadratic Equations

Method: for equations with real unknowns, separate real and imaginary parts and equate them. For the square root of $a + ib$, put $p + iq$, use $p^2 - q^2 = a$, $2pq = b$ and $p^2 + q^2 = \sqrt{a^2 + b^2}$.

Q4. If $x + iy = \sqrt{[(a + ib)/(c + id)]}$, prove that $(x^2 + y^2)^2 = (a^2 + b^2)/(c^2 + d^2)$. [6 Marks]

Solution:

1. Square both sides: $(x + iy)^2 = (a + ib)/(c + id)$.
2. Take the modulus of both sides and use $|z^2| = |z|^2$ and $|z_1/z_2| = |z_1|/|z_2|$: $|x + iy|^2 = |a + ib|/|c + id|$.
3. That is, $x^2 + y^2 = \sqrt{(a^2 + b^2)} / \sqrt{(c^2 + d^2)}$.
4. Square both sides: $(x^2 + y^2)^2 = (a^2 + b^2)/(c^2 + d^2)$.

Answer: Hence proved.

Q5. If z_1 and z_2 are non-zero complex numbers such that $|z_1 + z_2| = |z_1 - z_2|$, prove that $\arg z_1 - \arg z_2 = \pm\pi/2$. [6 Marks]

Solution:

1. Square the given condition: $|z_1 + z_2|^2 = |z_1 - z_2|^2$.
2. Using $|z|^2 = z\bar{z}$ and expanding (as in Q1): $|z_1|^2 + |z_2|^2 + 2\operatorname{Re}(z_1\bar{z}_2) = |z_1|^2 + |z_2|^2 - 2\operatorname{Re}(z_1\bar{z}_2)$.
3. So $4\operatorname{Re}(z_1\bar{z}_2) = 0$, i.e. $\operatorname{Re}(z_1\bar{z}_2) = 0$.
4. Hence $z_1\bar{z}_2$ is purely imaginary and non-zero, so $\arg(z_1\bar{z}_2) = \pm\pi/2$.
5. But $\arg(z_1\bar{z}_2) = \arg z_1 + \arg \bar{z}_2 = \arg z_1 - \arg z_2$.

Answer: $\arg z_1 - \arg z_2 = \pm\pi/2$ (proved).

Note: Geometrically: the diagonals of the parallelogram are equal, so it is a rectangle and the sides z_1 and z_2 are perpendicular.

Q6. Find the real values of x and y for which $(1 + i)y^2 + (6 + i) = (2 + i)x$. [6 Marks]

Solution:

1. Expand and group real and imaginary parts: $y^2 + iy^2 + 6 + i = 2x + ix$, i.e. $(y^2 + 6) + i(y^2 + 1) = 2x + ix$.
2. Equate real parts: $y^2 + 6 = 2x \dots(1)$.
3. Equate imaginary parts: $y^2 + 1 = x \dots(2)$.
4. Substitute $x = y^2 + 1$ from (2) in (1): $y^2 + 6 = 2(y^2 + 1) = 2y^2 + 2$.
5. So $y^2 = 4$, hence $y = \pm 2$.
6. Then $x = y^2 + 1 = 5$.

Answer: $x = 5$, $y = 2$ or $y = -2$

Note: Check: $(1 + i)(4) + 6 + i = 10 + 5i$ and $(2 + i)(5) = 10 + 5i$.

Q7. Solve the quadratic equation $z^2 - (3 - i)z + (4 - 3i) = 0$. [6 Marks]

Solution:

1. Discriminant: $D = (3 - i)^2 - 4(4 - 3i) = (9 - 6i - 1) - 16 + 12i = -8 + 6i$.
2. Find $\sqrt{D}$: let $\sqrt{-8 + 6i} = p + iq$. Then $p^2 - q^2 = -8$, $2pq = 6$ (so $pq = 3$), and $p^2 + q^2 = |-8 + 6i| = \sqrt{64 + 36} = 10$.
3. Add: $2p^2 = 2$, so $p = \pm 1$. Subtract: $2q^2 = 18$, so $q = \pm 3$. Since $pq = 3$ is positive, $\sqrt{D} = \pm(1 + 3i)$.
4. Use the formula $z = [(3 - i) \pm \sqrt{D}]/2 = [(3 - i) \pm (1 + 3i)]/2$.
5. $z = (3 - i + 1 + 3i)/2 = (4 + 2i)/2 = 2 + i$. $z = (3 - i - 1 - 3i)/2 = (2 - 4i)/2 = 1 - 2i$.

Answer: $z = 2 + i$ or $z = 1 - 2i$

Note: Check: sum of roots $= 3 - i$ and product $= (2 + i)(1 - 2i) = 4 - 3i$, matching the equation.

Type 3: Polar Form, Modulus, Argument and Powers

Method: write each number as $r \operatorname{cis} \theta$ after finding the quadrant. Then powers, products and quotients become simple operations on r and θ .

Q8. Find the modulus and argument of $z = 1 + \cos \alpha + i \sin \alpha$, where $-\pi < \alpha < \pi$. Hence find the modulus and argument of $(1 + \cos \alpha + i \sin \alpha)/(1 + \cos \alpha - i \sin \alpha)$. [6 Marks]

Solution:

1. Use $1 + \cos \alpha = 2\cos^2(\alpha/2)$ and $\sin \alpha = 2\sin(\alpha/2)\cos(\alpha/2)$.
2. $z = 2\cos^2(\alpha/2) + 2i \sin(\alpha/2)\cos(\alpha/2) = 2\cos(\alpha/2)[\cos(\alpha/2) + i \sin(\alpha/2)]$.
3. Since $-\pi/2 < \alpha/2 < \pi/2$, $\cos(\alpha/2) > 0$. So this is already the polar form with $r = 2\cos(\alpha/2)$ and $\theta = \alpha/2$.
4. Similarly, $w = 1 + \cos \alpha - i \sin \alpha = 2\cos(\alpha/2)[\cos(\alpha/2) - i \sin(\alpha/2)]$, with modulus $2\cos(\alpha/2)$ and argument $-\alpha/2$.
5. Quotient z/w : modulus $= 2\cos(\alpha/2)/2\cos(\alpha/2) = 1$, argument $= \alpha/2 - (-\alpha/2) = \alpha$.

Answer: $|z| = 2\cos(\alpha/2)$, $\arg z = \alpha/2$. For the quotient: modulus $= 1$, argument $= \alpha$, i.e. it equals $\cos \alpha + i \sin \alpha$.

Q9. Using polar form, evaluate (i) $(1 + i)^{12}/(1 - i)^8$ (ii) $[(\sqrt{3} + i)/(1 - i\sqrt{3})]^{10}$. [6 Marks]

Solution:

1. (i) $1 + i = \sqrt{2} \operatorname{cis}(\pi/4)$ and $1 - i = \sqrt{2} \operatorname{cis}(-\pi/4)$.
2. $(1 + i)^{12} = (\sqrt{2})^{12} \operatorname{cis}(12\pi/4) = 64 \operatorname{cis}(3\pi) = 64(-1) = -64$.
3. $(1 - i)^8 = (\sqrt{2})^8 \operatorname{cis}(-8\pi/4) = 16 \operatorname{cis}(-2\pi) = 16(1) = 16$.
4. So $(1 + i)^{12}/(1 - i)^8 = -64/16 = -4$.
5. (ii) $\sqrt{3} + i = 2\operatorname{cis}(\pi/6)$ and $1 - i\sqrt{3} = 2\operatorname{cis}(-\pi/3)$.
6. Quotient $= (2/2) \operatorname{cis}(\pi/6 + \pi/3) = \operatorname{cis}(\pi/2) = i$.
7. Then $[i]^{10} = \operatorname{cis}(10\pi/2) = \operatorname{cis}(5\pi) = \cos 5\pi + i \sin 5\pi = -1$.

Answer: (i) -4 (ii) -1

Note: Check (i) directly: $(1 + i)^2 = 2i$ so $(1 + i)^{12} = (2i)^6 = 64i^6 = -64$, and $(1 - i)^2 = -2i$ so $(1 - i)^8 = (-2i)^4 = 16$.

Type 4: De Moivre's Theorem and Euler's Form – Proofs

Method: in proofs, convert the given expression to cis form or $e^{i\theta}$ form, cancel common factors, and apply $x^n + 1/x^n = 2\cos n\theta$ or the Euler formulae $2\cos \theta = e^{i\theta} + e^{-i\theta}$, $2i \sin \theta = e^{i\theta} - e^{-i\theta}$.

Q10. If $x + 1/x = 2\cos \theta$, prove that $x^n + 1/x^n = 2\cos n\theta$. Hence, if $x + 1/x = 1$, find $x^{100} + 1/x^{100}$. [6 Marks]

Solution:

1. $x + 1/x = 2\cos \theta$ gives $x^2 - 2x \cos \theta + 1 = 0$, so $x = [2\cos \theta \pm \sqrt{(4\cos^2 \theta - 4)}]/2 = \cos \theta \pm i \sin \theta$.

2. Take $x = \cos\theta + i \sin\theta$. By De Moivre, $x^n = \cos n\theta + i \sin n\theta$ and $1/x^n = x^{-n} = \cos n\theta - i \sin n\theta$.
3. Add: $x^n + 1/x^n = 2\cos n\theta$. For $x = \cos\theta - i \sin\theta$ the result is the same, because cosine is an even function.
4. Now $x + 1/x = 1$ means $2\cos\theta = 1$, so $\cos\theta = 1/2$ and $\theta = \pi/3$.
5. $x^{100} + 1/x^{100} = 2\cos(100\pi/3)$. Write $100\pi/3 = 32\pi + 4\pi/3$, so $\cos(100\pi/3) = \cos(4\pi/3) = -1/2$.
6. Therefore $x^{100} + 1/x^{100} = 2(-1/2) = -1$.

Answer: Proved; $x^{100} + 1/x^{100} = -1$

Q11. If α and β are the roots of $x^2 - 2\sqrt{3}x + 4 = 0$, prove that $\alpha^n + \beta^n = 2^{n+1}\cos(n\pi/6)$. Hence find $\alpha^6 + \beta^6$ and $\alpha^{12} + \beta^{12}$. [6 Marks]

Solution:

1. Solve: $x = [2\sqrt{3} \pm \sqrt{(12 - 16)}]/2 = [2\sqrt{3} \pm 2i]/2 = \sqrt{3} \pm i$. Let $\alpha = \sqrt{3} + i$ and $\beta = \sqrt{3} - i$.
2. Polar form: $\alpha = 2(\cos\pi/6 + i \sin\pi/6)$ and $\beta = 2(\cos\pi/6 - i \sin\pi/6)$.
3. By De Moivre: $\alpha^n = 2^n(\cos n\pi/6 + i \sin n\pi/6)$ and $\beta^n = 2^n(\cos n\pi/6 - i \sin n\pi/6)$.
4. Add: $\alpha^n + \beta^n = 2^n \times 2\cos(n\pi/6) = 2^{n+1}\cos(n\pi/6)$.
5. $n = 6$: $2^7\cos\pi = 128 \times (-1) = -128$.
6. $n = 12$: $2^{13}\cos 2\pi = 8192 \times 1 = 8192$.

Answer: Proved; $\alpha^6 + \beta^6 = -128$, $\alpha^{12} + \beta^{12} = 8192$

Q12. Prove that $[(1 + i \tan\theta)/(1 - i \tan\theta)]^n = \cos 2n\theta + i \sin 2n\theta$. [6 Marks]

Solution:

1. Multiply numerator and denominator of the fraction by $\cos\theta$: $(1 + i \tan\theta)/(1 - i \tan\theta) = (\cos\theta + i \sin\theta)/(\cos\theta - i \sin\theta)$.
2. Since $1/(\cos\theta - i \sin\theta) = \cos\theta + i \sin\theta$, the fraction $= (\cos\theta + i \sin\theta)^2$.
3. By De Moivre: $(\cos\theta + i \sin\theta)^2 = \cos 2\theta + i \sin 2\theta$.
4. Raise to the power n : $(\cos 2\theta + i \sin 2\theta)^n = \cos 2n\theta + i \sin 2n\theta$.

Answer: Hence proved.

Q13. If $z = \cos\theta + i \sin\theta$ ($\theta \neq 0$), prove that $(1 + z)/(1 - z) = i \cot(\theta/2)$. [6 Marks]

Solution:

1. Use Euler's form $z = e^{i\theta}$.
2. $1 + z = 1 + e^{i\theta} = e^{i\theta/2}(e^{-i\theta/2} + e^{i\theta/2}) = e^{i\theta/2} \times 2\cos(\theta/2)$.
3. $1 - z = 1 - e^{i\theta} = e^{i\theta/2}(e^{-i\theta/2} - e^{i\theta/2}) = e^{i\theta/2} \times (-2i \sin(\theta/2))$.
4. Divide: $(1 + z)/(1 - z) = 2\cos(\theta/2)/[-2i \sin(\theta/2)] = \cot(\theta/2)/(-i)$.
5. Since $1/(-i) = i$, we get $(1 + z)/(1 - z) = i \cot(\theta/2)$.

Answer: Hence proved.

Type 5: Roots of Complex Numbers and Equations

Method: convert the number to polar form with $2k\pi$ added to the angle and apply the power $1/n$. For equations, either substitute (for example $w = z - 1$) or multiply by a suitable factor to get a binomial equation $x^n = a$.

Q14. Find all the values of $(1 - i\sqrt{3})^{1/3}$. [6 Marks]

Solution:

1. For $1 - i\sqrt{3}$: $r = \sqrt{1 + 3} = 2$. $\alpha = \tan^{-1}\sqrt{3} = \pi/3$ and the point $(1, -\sqrt{3})$ is in the fourth quadrant, so $\theta = -\pi/3$.
2. $1 - i\sqrt{3} = 2[\cos(-\pi/3 + 2k\pi) + i \sin(-\pi/3 + 2k\pi)]$.
3. $(1 - i\sqrt{3})^{1/3} = 2^{1/3}[\cos((- \pi/3 + 2k\pi)/3) + i \sin((- \pi/3 + 2k\pi)/3)] = 2^{1/3}[\cos(-\pi/9 + 2k\pi/3) + i \sin(-\pi/9 + 2k\pi/3)]$, $k = 0, 1, 2$.
4. $k = 0$: angle $-\pi/9$.
5. $k = 1$: angle $-\pi/9 + 2\pi/3 = 5\pi/9$.
6. $k = 2$: angle $-\pi/9 + 4\pi/3 = 11\pi/9$.

Answer: $2^{1/3}[\cos(\pi/9) - i \sin(\pi/9)]$, $2^{1/3}[\cos(5\pi/9) + i \sin(5\pi/9)]$, $2^{1/3}[\cos(11\pi/9) + i \sin(11\pi/9)]$

Note: The three roots lie on a circle of radius $2^{1/3}$ and are 120° apart.

Q15. Solve $x^4 - x^2 + 1 = 0$ using De Moivre's theorem. [6 Marks]

Solution:

1. Multiply the equation by $(x^2 + 1)$: $(x^2 + 1)(x^4 - x^2 + 1) = x^6 + 1$.
2. So the roots of the given equation are the roots of $x^6 + 1 = 0$, except those of $x^2 + 1 = 0$, i.e. except $x = \pm i$.
3. $x^6 = -1 = \cos(\pi + 2k\pi) + i \sin(\pi + 2k\pi)$, so $x = \cos((2k + 1)\pi/6) + i \sin((2k + 1)\pi/6)$, $k = 0, 1, 2, 3, 4, 5$.
4. The angles are $\pi/6, \pi/2, 5\pi/6, 7\pi/6, 3\pi/2, 11\pi/6$.
5. Angles $\pi/2$ and $3\pi/2$ give $x = i$ and $x = -i$, which are rejected.
6. Remaining roots: $\cos\pi/6 + i \sin\pi/6 = (\sqrt{3} + i)/2$; $\cos 5\pi/6 + i \sin 5\pi/6 = (-\sqrt{3} + i)/2$; $\cos 7\pi/6 + i \sin 7\pi/6 = (-\sqrt{3} - i)/2$; $\cos 11\pi/6 + i \sin 11\pi/6 = (\sqrt{3} - i)/2$.

Answer: $x = (\pm\sqrt{3} \pm i)/2$ (all four sign combinations)

Q16. Solve the equation $(z - 1)^3 = 8$. [6 Marks]

Solution:

1. Put $w = z - 1$. Then $w^3 = 8 = 8 \times 1$, so $w = 2 \times (\text{cube roots of unity}) = 2, 2\omega, 2\omega^2$, where $\omega = (-1 + i\sqrt{3})/2$.
2. $w = 2$, $w = 2\omega = -1 + i\sqrt{3}$, $w = 2\omega^2 = -1 - i\sqrt{3}$.
3. Since $z = 1 + w$: $z = 3$, $z = 1 + (-1 + i\sqrt{3}) = i\sqrt{3}$, $z = 1 + (-1 - i\sqrt{3}) = -i\sqrt{3}$.

Answer: $z = 3, i\sqrt{3}, -i\sqrt{3}$

Note: Check: $(z - 1)^3 - 8 = z^3 - 3z^2 + 3z - 9 = (z - 3)(z^2 + 3)$, whose roots are 3 and $\pm i\sqrt{3}$.

Type 6: Cube Roots of Unity

Method: use $\omega^3 = 1$, $1 + \omega + \omega^2 = 0$, $1 + \omega = -\omega^2$ and $1 + \omega^2 = -\omega$. Reduce large powers by dividing the index by 3.

Q17. If ω is a complex cube root of unity, prove that (i) $(1 + \omega)(1 + \omega^2)(1 + \omega^4)(1 + \omega^8) = 1$ (ii) $1/(1 + \omega) + 1/(1 + \omega^2) = 1$. [6 Marks]

Solution:

1. Recall $\omega^3 = 1$, $1 + \omega = -\omega^2$, $1 + \omega^2 = -\omega$. Also $\omega^4 = \omega$ and $\omega^8 = \omega^2$ (since $8 = 3 \times 2 + 2$).
2. (i) The product $= (1 + \omega)(1 + \omega^2)(1 + \omega)(1 + \omega^2) = [(1 + \omega)(1 + \omega^2)]^2$.
3. $(1 + \omega)(1 + \omega^2) = (-\omega^2)(-\omega) = \omega^3 = 1$. So the product $= 1^2 = 1$.
4. (ii) $1/(1 + \omega) = 1/(-\omega^2) = -\omega^{-2} = -\omega$ (as $\omega^{-2} = \omega$).
5. $1/(1 + \omega^2) = 1/(-\omega) = -\omega^{-1} = -\omega^2$.
6. Sum $= -\omega - \omega^2 = -(\omega + \omega^2) = -(-1) = 1$.

Answer: (i) 1 (ii) 1 (both proved)

Q18. If ω is a complex cube root of unity, prove that (i) $(3 + 3\omega + 5\omega^2)^6 - (2 + 6\omega + 2\omega^2)^3 = 0$ (ii) if α and β are the complex cube roots of unity then $\alpha^4 + \beta^4 + \alpha^{-1}\beta^{-1} = 0$. [6 Marks]

Solution:

1. (i) $3 + 3\omega + 5\omega^2 = 3(1 + \omega) + 5\omega^2 = 3(-\omega^2) + 5\omega^2 = 2\omega^2$.
2. $2 + 6\omega + 2\omega^2 = 2(1 + \omega + \omega^2) + 4\omega = 0 + 4\omega = 4\omega$.
3. $(2\omega^2)^6 = 64\omega^{12} = 64(\omega^3)^4 = 64$. $(4\omega)^3 = 64\omega^3 = 64$. So the difference $= 64 - 64 = 0$.
4. (ii) The complex cube roots of unity are ω and ω^2 . Take $\alpha = \omega$ and $\beta = \omega^2$ (interchanging them gives the same result).
5. $\alpha^4 = \omega^4 = \omega$. $\beta^4 = \omega^8 = \omega^2$. $\alpha^{-1}\beta^{-1} = 1/(\alpha\beta) = 1/\omega^3 = 1$.
6. Sum $= \omega + \omega^2 + 1 = 0$.

Answer: Both results proved.

Type 7: Expansion of $\cos n\theta$ and $\tan n\theta$

Method: expand $(\cos\theta + i \sin\theta)^n$ by the binomial theorem, using $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, and equate real and imaginary parts. Replace $\sin^2\theta$ by $1 - \cos^2\theta$ to get $\cos n\theta$ in powers of $\cos\theta$. For $\tan n\theta$ divide by $\cos^n\theta$.

Q19. Prove that $\tan 5\theta = (5\tan\theta - 10\tan^3\theta + \tan^5\theta)/(1 - 10\tan^2\theta + 5\tan^4\theta)$. [6 Marks]

Solution:

1. Let $c = \cos\theta$, $s = \sin\theta$. $\cos 5\theta + i \sin 5\theta = (c + is)^5 = c^5 + 5ic^4s - 10c^3s^2 - 10ic^2s^3 + 5cs^4 + is^5$.
2. Real part: $\cos 5\theta = c^5 - 10c^3s^2 + 5cs^4$. Imaginary part: $\sin 5\theta = 5c^4s - 10c^2s^3 + s^5$.
3. $\tan 5\theta = \sin 5\theta / \cos 5\theta$. Divide the numerator and the denominator by c^5 , and put $t = \tan\theta = s/c$.
4. Numerator: $(5c^4s - 10c^2s^3 + s^5)/c^5 = 5t - 10t^3 + t^5$.

5. Denominator: $(c^5 - 10c^3s^2 + 5cs^4)/c^5 = 1 - 10t^2 + 5t^4$.

Answer: $\tan 5\theta = (5t - 10t^3 + t^5)/(1 - 10t^2 + 5t^4)$, where $t = \tan\theta$ (proved)

Note: Check with $\theta = 45^\circ$ ($t = 1$): numerator = $5 - 10 + 1 = -4$, denominator = $1 - 10 + 5 = -4$, so $\tan 225^\circ = 1$, which is correct.

Q20. Prove that $\cos 7\theta = 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta$. [6 Marks]

Solution:

1. Let $c = \cos\theta$, $s = \sin\theta$. $\cos 7\theta + i \sin 7\theta = (c + is)^7$.
2. The real terms of the binomial expansion are those with even powers of i : $c^7 + 21c^5(is)^2 + 35c^3(is)^4 + 7c(is)^6$.
3. Since $i^2 = -1$, $i^4 = 1$, $i^6 = -1$: $\cos 7\theta = c^7 - 21c^5s^2 + 35c^3s^4 - 7cs^6$.
4. Put $s^2 = 1 - c^2$: $-21c^5(1 - c^2) = -21c^5 + 21c^7$; $35c^3(1 - c^2)^2 = 35c^3 - 70c^5 + 35c^7$; $-7c(1 - c^2)^3 = -7c + 21c^3 - 21c^5 + 7c^7$.
5. Add all terms with c^7 : $1 + 21 + 35 + 7 = 64$. c^5 : $-21 - 70 - 21 = -112$. c^3 : $35 + 21 = 56$. c : -7 .

Answer: $\cos 7\theta = 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta$ (proved)

Note: Check with $\theta = 0$: $64 - 112 + 56 - 7 = 1 = \cos 0$.

Type 8: Powers of $\sin\theta$ and $\cos\theta$ in Multiple Angles

Method: put $z = \cos\theta + i \sin\theta$, so $2\cos\theta = z + 1/z$ and $2i \sin\theta = z - 1/z$. For mixed powers, use $(z + 1/z)(z - 1/z) = z^2 - 1/z^2$ to reduce the work. Finally use $z^n + 1/z^n = 2\cos n\theta$.

Q21. Prove that $\cos^6\theta = (1/32)(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10)$. [6 Marks]

Solution:

1. Let $z = \cos\theta + i \sin\theta$, so $2\cos\theta = z + 1/z$. Then $(2\cos\theta)^6 = (z + 1/z)^6$, and LHS = $64\cos^6\theta$.
2. Expand: $(z + 1/z)^6 = z^6 + 6z^4 + 15z^2 + 20 + 15/z^2 + 6/z^4 + 1/z^6$.
3. Group in pairs: $= (z^6 + 1/z^6) + 6(z^4 + 1/z^4) + 15(z^2 + 1/z^2) + 20 = 2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20$.
4. So $64\cos^6\theta = 2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20$. Divide by 64.

Answer: $\cos^6\theta = (1/32)(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10)$

Note: Check with $\theta = 0$: LHS = 1 and RHS = $(1 + 6 + 15 + 10)/32 = 1$.

Q22. Express $\sin^4\theta \cos^2\theta$ in a series of cosines of multiples of θ . [6 Marks]

Solution:

1. Let $z = \cos\theta + i \sin\theta$. Then $(2i \sin\theta)^4(2\cos\theta)^2 = (z - 1/z)^4(z + 1/z)^2$.
2. LHS = $16\sin^4\theta \times 4\cos^2\theta = 64 \sin^4\theta \cos^2\theta$ (since $i^4 = 1$).
3. RHS: $(z - 1/z)^4(z + 1/z)^2 = [(z - 1/z)(z + 1/z)]^2(z - 1/z)^2 = (z^2 - 1/z^2)^2(z - 1/z)^2 = (z^4 - 2 + 1/z^4)(z^2 - 2 + 1/z^2)$.
4. Multiply term by term: $z^6 - 2z^4 + z^2 - 2z^2 + 4 - 2/z^2 + 1/z^2 - 2/z^4 + 1/z^6 = z^6 - 2z^4 - z^2 + 4 - 1/z^2 - 2/z^4 + 1/z^6$.

5. Group in pairs: $(z^6 + 1/z^6) - 2(z^4 + 1/z^4) - (z^2 + 1/z^2) + 4 = 2\cos 6\theta - 4\cos 4\theta - 2\cos 2\theta + 4$.
 6. So $64 \sin^4 \theta \cos^2 \theta = 2\cos 6\theta - 4\cos 4\theta - 2\cos 2\theta + 4$. Divide by 64.

Answer: $\sin^4 \theta \cos^2 \theta = (1/32)(\cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2)$

Note: Check with $\theta = \pi/4$: $LHS = (1/4)(1/2) = 1/8$ and $RHS = (1/32)(0 + 2 - 0 + 2) = 1/8$.

Type 9: Locus in the Argand Plane

Method: put $z = x + iy$, write the given condition in terms of x and y , and simplify to get the equation of a circle, line, ray or other curve. For an argument condition, also note the sign of the real and imaginary parts.

Q23. Find and describe the locus of the point $z = x + iy$ in each case: (i) $|z - 1 - 2i| = 3$ (ii) $\operatorname{Re}(z^2) = 4$ (iii) $\arg(z - 1) = \pi/4$. [6 Marks]

Solution:

- (i) $z - 1 - 2i = (x - 1) + i(y - 2)$. Its modulus is 3, so $(x - 1)^2 + (y - 2)^2 = 9$.
- (ii) $z^2 = (x + iy)^2 = (x^2 - y^2) + 2ixy$. So $\operatorname{Re}(z^2) = x^2 - y^2 = 4$, i.e. $x^2/4 - y^2/4 = 1$.
- (iii) $z - 1 = (x - 1) + iy$. An argument of $\pi/4$ means $y/(x - 1) = \tan(\pi/4) = 1$, with $x - 1 > 0$ and $y > 0$ (first quadrant).
- So $y = x - 1$ with $x > 1$.

Answer: (i) Circle with centre (1, 2) and radius 3. (ii) Rectangular hyperbola $x^2 - y^2 = 4$ with vertices $(\pm 2, 0)$. (iii) The half-line $y = x - 1$, $x > 1$, starting from the point (1, 0) but not including it.