

MATHEMATICS 1

Chapter: Complex Numbers

Medium Answer Questions with Solutions | 4 Marks | All Types

This document contains the most important medium answer (4 marks) questions of the chapter Complex Numbers. They are arranged type-wise, and every question is solved in clear steps with a final answer. These are the questions that fall between the short (2 marks) and the broad (6 marks) questions, and a typical exam paper carries several of them. All questions are new and different from the earlier documents.

Types covered:

- Type 1: Algebra and properties of complex numbers (M1–M4)
- Type 2: Polar and exponential form (M5–M7)
- Type 3: De Moivre's theorem (M8–M10)
- Type 4: Roots of complex numbers and equations (M11–M13)
- Type 5: Cube roots of unity (M14–M15)
- Type 6: Powers of $\sin\theta$ and $\cos\theta$ in multiple angles (M16–M17)
- Type 7: Locus and geometry in the Argand plane (M18–M21)

Type 1: Algebra and Properties of Complex Numbers

Method: rationalise using the conjugate, separate real and imaginary parts, and use $|z_1 z_2| = |z_1| |z_2|$, $z\bar{z} = |z|^2$ wherever they shorten the work.

M1. Express $(2 + i)/(1 - i) + (3 - i)/(1 + i)$ in the form $a + ib$. [4 Marks]

Solution:

1. First term: $(2 + i)/(1 - i) = (2 + i)(1 + i)/[(1 - i)(1 + i)] = (2 + 2i + i + i^2)/2 = (1 + 3i)/2$.
2. Second term: $(3 - i)/(1 + i) = (3 - i)(1 - i)/[(1 + i)(1 - i)] = (3 - 3i - i + i^2)/2 = (2 - 4i)/2 = 1 - 2i$.
3. Add: $(1/2 + 3i/2) + (1 - 2i) = (1/2 + 1) + i(3/2 - 2)$.

Answer: $3/2 - (1/2)i$

M2. Find the real numbers x and y if $(x + 3i)(2 - iy) = 7 + 11i$. [4 Marks]

Solution:

1. Expand: $(x + 3i)(2 - iy) = 2x - ixy + 6i - 3i^2y = (2x + 3y) + i(6 - xy)$.
2. Equate real parts: $2x + 3y = 7$... (1). Equate imaginary parts: $6 - xy = 11$, so $xy = -5$... (2).
3. From (1), $x = (7 - 3y)/2$. Put in (2): $y(7 - 3y)/2 = -5$, i.e. $3y^2 - 7y - 10 = 0$, which factorises as $(3y + 10)(y - 1) = 0$.
4. So $y = -1$ or $y = 10/3$. Then $x = (7 + 3)/2 = 5$ or $x = (7 - 10)/2 = -3/2$.

Answer: $(x, y) = (5, -1)$ or $(x, y) = (-3/2, 10/3)$

Note: Check: $5 \times (-1) = -5$ and $(-3/2)(10/3) = -5$.

M3. If $z_1 = 3 + 4i$ and $z_2 = 5 - 12i$, verify that $|z_1 z_2| = |z_1||z_2|$ and $|z_1/z_2| = |z_1|/|z_2|$. [4 Marks]

Solution:

1. $|z_1| = \sqrt{9 + 16} = 5$ and $|z_2| = \sqrt{25 + 144} = 13$.
2. $z_1 z_2 = (3 + 4i)(5 - 12i) = 15 - 36i + 20i - 48i^2 = 63 - 16i$. So $|z_1 z_2| = \sqrt{3969 + 256} = \sqrt{4225} = 65 = 5 \times 13 = |z_1||z_2|$.
1. $z_1/z_2 = (3 + 4i)(5 + 12i)/(25 + 144) = (15 + 36i + 20i + 48i^2)/169 = (-33 + 56i)/169$.
2. $|z_1/z_2| = \sqrt{(1089 + 3136)/169} = 65/169 = 5/13 = |z_1|/|z_2|$.

Answer: Both results verified: $65 = 5 \times 13$ and $5/13 = 5/13$.

M4. If $|z| = 1$ and $z \neq -1$, prove that $(z - 1)/(z + 1)$ is purely imaginary (or zero). [4 Marks]

Solution:

1. Let $w = (z - 1)/(z + 1)$. Since $|z| = 1$, $z\bar{z} = 1$, so $\bar{z} = 1/z$.
2. Conjugate of w : $\bar{w} = (\bar{z} - 1)/(\bar{z} + 1) = (1/z - 1)/(1/z + 1)$.
3. Multiply the numerator and the denominator by z : $\bar{w} = (1 - z)/(1 + z) = -(z - 1)/(z + 1) = -w$.
4. So $w + \bar{w} = 0$, i.e. $2\text{Re}(w) = 0$. Hence the real part of w is zero.

Answer: w is purely imaginary (it is 0 when $z = 1$). Hence proved.

Type 2: Polar and Exponential Form

Method: find r and θ (checking the quadrant), write $z = r \text{cis} \theta = re^{i\theta}$, then use De Moivre for powers and the Euler formulae $e^{i\theta} \pm e^{-i\theta}$ for simplification.

M5. Express $(\sqrt{3} + i)/(\sqrt{3} - i)$ in polar form and hence in the form $a + ib$. [4 Marks]

Solution:

1. $\sqrt{3} + i$: $r = 2$, $\theta = \tan^{-1}(1/\sqrt{3}) = \pi/6$. $\sqrt{3} - i$: $r = 2$, $\theta = -\pi/6$ (fourth quadrant).
2. Quotient: modulus $= 2/2 = 1$, argument $= \pi/6 - (-\pi/6) = \pi/3$.
3. Polar form: $\cos(\pi/3) + i \sin(\pi/3) = 1/2 + i\sqrt{3}/2$.
4. Check by rationalising: $(\sqrt{3} + i)^2/(3 + 1) = (3 - 1 + 2\sqrt{3}i)/4 = (1 + \sqrt{3}i)/2$.

Answer: $\cos(\pi/3) + i \sin(\pi/3) = 1/2 + (\sqrt{3}/2)i$

M6. Express $-2 + 2i$ in polar form and hence find $(-2 + 2i)^6$. [4 Marks]

Solution:

1. $r = \sqrt{4 + 4} = 2\sqrt{2}$. $\alpha = \tan^{-1}(2/2) = \pi/4$. The point $(-2, 2)$ is in the second quadrant, so $\theta = \pi - \pi/4 = 3\pi/4$.
2. Polar form: $-2 + 2i = 2\sqrt{2} (\cos 3\pi/4 + i \sin 3\pi/4)$.
3. By De Moivre: $(-2 + 2i)^6 = (2\sqrt{2})^6 [\cos(18\pi/4) + i \sin(18\pi/4)] = 512[\cos(9\pi/2) + i \sin(9\pi/2)]$.

4. $9\pi/2 = 4\pi + \pi/2$, so the bracket $= \cos(\pi/2) + i \sin(\pi/2) = i$.

Answer: $-2 + 2i = 2\sqrt{2} (\cos 3\pi/4 + i \sin 3\pi/4)$; $(-2 + 2i)^6 = 512i$

Note: Check: $(-2 + 2i)^2 = -8i$ and $(-8i)^3 = -512i^3 = 512i$.

M7. Using Euler's formula, prove that $(e^{i\theta} - e^{-i\theta})/(e^{i\theta} + e^{-i\theta}) = i \tan \theta$. Also find the value of $e^{i\pi/3} + e^{-i\pi/3}$. [4 Marks]

Solution:

1. Euler's formula: $e^{i\theta} = \cos \theta + i \sin \theta$ and $e^{-i\theta} = \cos \theta - i \sin \theta$.
2. Subtract: $e^{i\theta} - e^{-i\theta} = 2i \sin \theta$. Add: $e^{i\theta} + e^{-i\theta} = 2 \cos \theta$.
3. Ratio $= 2i \sin \theta / (2 \cos \theta) = i \tan \theta$.
4. Put $\theta = \pi/3$ in the sum formula: $e^{i\pi/3} + e^{-i\pi/3} = 2 \cos(\pi/3) = 2 \times 1/2 = 1$.

Answer: Ratio $= i \tan \theta$ (proved); $e^{i\pi/3} + e^{-i\pi/3} = 1$

Type 3: De Moivre's Theorem

Method: write every factor as $\cos n\theta + i \sin n\theta$ (using $\cos n\theta - i \sin n\theta = \cos(-n\theta) + i \sin(-n\theta)$), apply the index laws, and for proofs use $x^n + 1/x^n = 2 \cos n\theta$ or the Euler form.

M8. Simplify $[(\cos 3\theta + i \sin 3\theta)^2 (\cos \theta - i \sin \theta)^{-3}] / (\cos 2\theta + i \sin 2\theta)^{-4}$. [4 Marks]

Solution:

1. Let $\text{cis } \phi = \cos \phi + i \sin \phi$, and note $\cos \theta - i \sin \theta = \text{cis}(-\theta)$.
2. $(\cos 3\theta + i \sin 3\theta)^2 = \text{cis } 6\theta$. $(\cos \theta - i \sin \theta)^{-3} = [\text{cis}(-\theta)]^{-3} = \text{cis } 3\theta$. So the numerator $= \text{cis } 6\theta \times \text{cis } 3\theta = \text{cis } 9\theta$.
3. Denominator: $(\cos 2\theta + i \sin 2\theta)^{-4} = \text{cis}(-8\theta)$.
4. Ratio $= \text{cis } 9\theta / \text{cis}(-8\theta) = \text{cis}(9\theta + 8\theta) = \text{cis } 17\theta$.

Answer: $\cos 17\theta + i \sin 17\theta$

M9. If $z = \cos \theta + i \sin \theta$ and $w = \cos \phi + i \sin \phi$, prove that $(z + w)/(1 + zw) = \cos[(\theta - \phi)/2] / \cos[(\theta + \phi)/2]$. [4 Marks]

Solution:

1. Use Euler's form: $z = e^{i\theta}$ and $w = e^{i\phi}$.
2. $z + w = e^{i(\theta+\phi)/2} [e^{i(\theta-\phi)/2} + e^{-i(\theta-\phi)/2}] = e^{i(\theta+\phi)/2} \times 2 \cos[(\theta - \phi)/2]$.
3. $1 + zw = 1 + e^{i(\theta+\phi)} = e^{i(\theta+\phi)/2} [e^{-i(\theta+\phi)/2} + e^{i(\theta+\phi)/2}] = e^{i(\theta+\phi)/2} \times 2 \cos[(\theta + \phi)/2]$.
4. Divide: the common factor $e^{i(\theta+\phi)/2}$ and the factor 2 cancel, leaving $\cos[(\theta - \phi)/2] / \cos[(\theta + \phi)/2]$.

Answer: Hence proved.

Note: Check with $\theta = \pi/3$, $\phi = 0$: $z = 1/2 + i\sqrt{3}/2$, $w = 1$, and $(z + 1)/(1 + z) = 1$, while the right side is $\cos(\pi/6)/\cos(\pi/6) = 1$.

M10. If $z + 1/z = \sqrt{3}$, find $z^6 + 1/z^6$ and $z^{12} + 1/z^{12}$. [4 Marks]

Solution:

1. $z + 1/z = \sqrt{3}$ gives $z^2 - \sqrt{3}z + 1 = 0$, so $z = [\sqrt{3} \pm \sqrt{(3-4)}]/2 = (\sqrt{3} \pm i)/2 = \cos(\pi/6) \pm i \sin(\pi/6)$.
2. Take $z = \cos\pi/6 + i \sin\pi/6$. By De Moivre, $z^n + 1/z^n = 2\cos(n\pi/6)$.
3. $n = 6$: $2\cos\pi = -2$.
4. $n = 12$: $2\cos 2\pi = 2$.

Answer: $z^6 + 1/z^6 = -2$, $z^{12} + 1/z^{12} = 2$

Type 4: Roots of Complex Numbers and Equations

Method: write the number in polar form as $r[\cos(\theta + 2k\pi) + i \sin(\theta + 2k\pi)]$ and use the n th root formula with $k = 0$ to $n - 1$. For an equation like $(z + a)^n = b$, substitute $w = z + a$ first.

M11. Find the cube roots of -8 and express them in the form $a + ib$. [4 Marks]

Solution:

1. $x^3 = -8 = 8[\cos(\pi + 2k\pi) + i \sin(\pi + 2k\pi)]$.
2. $x = 8^{1/3}[\cos((2k+1)\pi/3) + i \sin((2k+1)\pi/3)] = 2[\cos((2k+1)\pi/3) + i \sin((2k+1)\pi/3)]$, $k = 0, 1, 2$.
3. $k = 0$: $2(\cos\pi/3 + i \sin\pi/3) = 1 + i\sqrt{3}$. $k = 1$: $2(\cos\pi + i \sin\pi) = -2$. $k = 2$: $2(\cos 5\pi/3 + i \sin 5\pi/3) = 1 - i\sqrt{3}$.

Answer: $1 + i\sqrt{3}$, -2 , $1 - i\sqrt{3}$

Note: The sum of the three roots is $1 - 2 + 1 = 0$, as expected.

M12. Solve the equation $(z + 1)^4 = 16$. [4 Marks]

Solution:

1. Put $w = z + 1$. Then $w^4 = 16 = 16(\cos 2k\pi + i \sin 2k\pi)$.
2. $w = 16^{1/4}[\cos(2k\pi/4) + i \sin(2k\pi/4)] = 2[\cos(k\pi/2) + i \sin(k\pi/2)]$, $k = 0, 1, 2, 3$.
3. $w = 2, 2i, -2, -2i$.
4. Since $z = w - 1$: $z = 1, -1 + 2i, -3, -1 - 2i$.

Answer: $z = 1, -1 + 2i, -3, -1 - 2i$

M13. Find all the values of $(1 + i\sqrt{3})^{1/3}$ and show that the product of the three values is $1 + i\sqrt{3}$. [4 Marks]

Solution:

1. $1 + i\sqrt{3}$: $r = 2$, $\theta = \pi/3$. So $1 + i\sqrt{3} = 2[\cos(\pi/3 + 2k\pi) + i \sin(\pi/3 + 2k\pi)]$.
2. $(1 + i\sqrt{3})^{1/3} = 2^{1/3}[\cos(\pi/9 + 2k\pi/3) + i \sin(\pi/9 + 2k\pi/3)]$, $k = 0, 1, 2$.
3. The arguments are $k = 0$: $\pi/9$; $k = 1$: $\pi/9 + 6\pi/9 = 7\pi/9$; $k = 2$: $\pi/9 + 12\pi/9 = 13\pi/9$.
4. Product: modulus $= (2^{1/3})^3 = 2$; argument $= \pi/9 + 7\pi/9 + 13\pi/9 = 21\pi/9 = 7\pi/3 = 2\pi + \pi/3$.

5. So product = $2(\cos\pi/3 + i \sin\pi/3) = 1 + i\sqrt{3}$.

Answer: Values: $2^{1/3}\text{cis}(\pi/9)$, $2^{1/3}\text{cis}(7\pi/9)$, $2^{1/3}\text{cis}(13\pi/9)$; product = $1 + i\sqrt{3}$ (verified)

Type 5: Cube Roots of Unity

Method: use $\omega^3 = 1$ and $1 + \omega + \omega^2 = 0$. Replace $1 + \omega$ by $-\omega^2$, and reduce every large power of ω to ω^0 , ω^1 or ω^2 .

M14. If ω is a cube root of unity, find the value of $(1 + 5\omega^2 + \omega^4)(1 + 5\omega + \omega^2)(5 + \omega + \omega^2)$.
[4 Marks]

Solution:

1. $\omega^4 = \omega^3 \cdot \omega = \omega$, so the first bracket = $1 + 5\omega^2 + \omega = (1 + \omega + \omega^2) + 4\omega^2 = 4\omega^2$.
2. Second bracket = $(1 + \omega + \omega^2) + 4\omega = 4\omega$.
3. Third bracket = $4 + (1 + \omega + \omega^2) = 4$.
4. Product = $4\omega^2 \times 4\omega \times 4 = 64\omega^3 = 64$.

Answer: 64

M15. If ω is a complex cube root of unity and n is a positive integer, prove that $\omega^n + \omega^{2n} = 2$ when n is a multiple of 3, and $\omega^n + \omega^{2n} = -1$ otherwise. [4 Marks]

Solution:

1. Case 1: $n = 3k$. $\omega^{3k} = 1$ and $\omega^{6k} = 1$, so $\omega^n + \omega^{2n} = 1 + 1 = 2$.
2. Case 2: $n = 3k + 1$. $\omega^n = \omega$ and $\omega^{2n} = \omega^{6k+2} = \omega^2$, so the sum = $\omega + \omega^2 = -1$.
3. Case 3: $n = 3k + 2$. $\omega^n = \omega^2$ and $\omega^{2n} = \omega^{6k+4} = \omega^4 = \omega$, so the sum = $\omega^2 + \omega = -1$.
4. Every positive integer falls in one of these three cases.

Answer: $\omega^n + \omega^{2n} = 2$ if $3 \mid n$, and $= -1$ otherwise (proved).

Type 6: Powers of $\sin\theta$ and $\cos\theta$ in Multiple Angles

Method: put $z = \cos\theta + i \sin\theta$ so that $2\cos\theta = z + 1/z$ and $2i \sin\theta = z - 1/z$. Expand, group the terms as $z^n \pm 1/z^n$, and replace them by $2\cos n\theta$ or $2i \sin n\theta$.

M16. Prove that $\sin^2\theta \cos^2\theta = (1/8)(1 - \cos 4\theta)$. [4 Marks]

Solution:

1. Let $z = \cos\theta + i \sin\theta$. Then $(2i \sin\theta)^2(2\cos\theta)^2 = (z - 1/z)^2(z + 1/z)^2$.
2. LHS = $(-4\sin^2\theta)(4\cos^2\theta) = -16 \sin^2\theta \cos^2\theta$.
3. RHS = $[(z - 1/z)(z + 1/z)]^2 = (z^2 - 1/z^2)^2 = z^4 - 2 + 1/z^4 = 2\cos 4\theta - 2$.
4. So $-16 \sin^2\theta \cos^2\theta = 2\cos 4\theta - 2$, i.e. $\sin^2\theta \cos^2\theta = (2 - 2\cos 4\theta)/16$.

Answer: $\sin^2\theta \cos^2\theta = (1/8)(1 - \cos 4\theta)$ (proved)

Note: Check with $\theta = \pi/4$: LHS = $(1/2)(1/2) = 1/4$ and RHS = $(1 - \cos\pi)/8 = 2/8 = 1/4$.

M17. Express $\sin^4\theta$ in a series of cosines of multiples of θ . [4 Marks]

Solution:

1. Let $z = \cos\theta + i \sin\theta$. Then $(2i \sin\theta)^4 = (z - 1/z)^4$. LHS = $16i^4 \sin^4\theta = 16 \sin^4\theta$.
2. Expand: $(z - 1/z)^4 = z^4 - 4z^2 + 6 - 4/z^2 + 1/z^4 = (z^4 + 1/z^4) - 4(z^2 + 1/z^2) + 6$.
3. $= 2\cos 4\theta - 8\cos 2\theta + 6$.
4. So $16 \sin^4\theta = 2\cos 4\theta - 8\cos 2\theta + 6$. Divide by 16.

Answer: $\sin^4\theta = (1/8)(\cos 4\theta - 4\cos 2\theta + 3)$

Note: Check with $\theta = \pi/2$: LHS = 1 and RHS = $(1 + 4 + 3)/8 = 1$.

Type 7: Locus and Geometry in the Argand Plane

Method: put $z = x + iy$, translate the condition into an equation in x and y , and identify the curve. For geometry, use the distance $|z_1 - z_2|$ between two points.

M18. Find the locus of the point z if $z + \bar{z} = |z|^2$. [4 Marks]

Solution:

1. Put $z = x + iy$. Then $z + \bar{z} = 2x$ and $|z|^2 = x^2 + y^2$.
2. So $x^2 + y^2 = 2x$, i.e. $x^2 - 2x + y^2 = 0$.
3. Complete the square: $(x - 1)^2 + y^2 = 1$.

Answer: A circle with centre (1, 0) and radius 1 (it passes through the origin).

M19. Find the equation of the locus of z if $|z - 3| = |z - 5i|$. [4 Marks]

Solution:

1. Put $z = x + iy$ and square both sides: $(x - 3)^2 + y^2 = x^2 + (y - 5)^2$.
2. Expand: $x^2 - 6x + 9 + y^2 = x^2 + y^2 - 10y + 25$.
3. Simplify: $-6x + 9 = -10y + 25$, i.e. $3x - 5y + 8 = 0$.

Answer: The straight line $3x - 5y + 8 = 0$ (the perpendicular bisector of the points 3 and 5i)

Note: Check: the midpoint $(3/2, 5/2)$ of the two points satisfies $3(3/2) - 5(5/2) + 8 = 0$.

M20. Show that the points $1 + i$, $-1 + i$, $-1 - i$ and $1 - i$ are the vertices of a square. [4 Marks]

Solution:

1. Let $A = 1 + i$, $B = -1 + i$, $C = -1 - i$, $D = 1 - i$.
2. Sides: $AB = |B - A| = |-2| = 2$; $BC = |C - B| = |-2i| = 2$; $CD = |D - C| = |2| = 2$; $DA = |A - D| = |2i| = 2$.
3. Diagonals: $AC = |C - A| = |-2 - 2i| = 2\sqrt{2}$; $BD = |D - B| = |2 - 2i| = 2\sqrt{2}$.
4. All four sides are equal and the two diagonals are equal, so ABCD is a square.

Answer: Proved: ABCD is a square of side 2.

M21. Show that the points 1, ω and ω^2 (the cube roots of unity) are the vertices of an equilateral triangle. [4 Marks]

Solution:

1. Take $\omega = (-1 + i\sqrt{3})/2$ and $\omega^2 = (-1 - i\sqrt{3})/2$.
2. $|1 - \omega| = |(3 - i\sqrt{3})/2| = \sqrt{[(9 + 3)/4]} = \sqrt{3}$.
3. $|\omega - \omega^2| = |i\sqrt{3}| = \sqrt{3}$.
4. $|\omega^2 - 1| = |(-3 - i\sqrt{3})/2| = \sqrt{[(9 + 3)/4]} = \sqrt{3}$.
5. All three sides are equal to $\sqrt{3}$.

Answer: Proved: the triangle is equilateral with side $\sqrt{3}$, inscribed in the unit circle.

Exam Tips for 4-Mark Questions

- A 4-mark question usually needs 4 to 6 lines. Write the formula, the substitution, the simplification and the answer.
- In verification questions, show both sides separately and state clearly that they are equal.
- For equations with real unknowns x and y, always equate real parts and imaginary parts separately.
- For roots, list every value of k. Missing even one root loses marks.