

MATHEMATICS 1

Chapter: Complex Numbers

Important Questions with Solutions | Exam-Oriented Exercise Notes for Engineering Students

These notes contain the most important questions of the chapter Complex Numbers, arranged from short to long, each with step-by-step solutions and the final answer. Revise the key formulae first, then practise the questions section by section.

Section 1: Key Points and Formulae (Quick Revision)

Section A: Short Answer Questions (2 Marks) – 10 questions

Section B: Medium Answer Questions (4 Marks) – 14 questions

Section C: Long Answer Questions (6 Marks) – 9 questions

Section D: Practice Questions with Answers, and Exam Tips

Section 1: Key Points and Formulae

- **Imaginary unit:** $i = \sqrt{-1}$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$. To find i^n , divide n by 4 and use the remainder r : $i^n = i^r$. Also $1/i = -i$.
- **Complex number:** $z = a + ib$, where $a = \text{Re}(z)$ is the real part and $b = \text{Im}(z)$ is the imaginary part. Conjugate: $\bar{z} = a - ib$.
- **Equality:** $a + ib = c + id \Leftrightarrow a = c$ and $b = d$.
- **Algebra:** $(a+ib) + (c+id) = (a+c) + i(b+d)$; $(a+ib)(c+id) = (ac - bd) + i(ad + bc)$. For division, multiply numerator and denominator by the conjugate of the denominator.
- **Modulus:** $|z| = \sqrt{a^2 + b^2}$. $z \cdot \bar{z} = |z|^2$; $|z_1 z_2| = |z_1| |z_2|$; $|z_1/z_2| = |z_1|/|z_2|$; $|z^n| = |z|^n$.
- **Argument:** Let $\alpha = \tan^{-1}|b/a|$ (acute angle). Quadrant I ($a > 0$, $b > 0$): $\theta = \alpha$. Quadrant II ($a < 0$, $b > 0$): $\theta = \pi - \alpha$. Quadrant III ($a < 0$, $b < 0$): $\theta = -(\pi - \alpha)$. Quadrant IV ($a > 0$, $b < 0$): $\theta = -\alpha$. Principal value lies in $-\pi < \theta \leq \pi$.
- **Properties of argument:** $\arg(z_1 z_2) = \arg z_1 + \arg z_2$; $\arg(z_1/z_2) = \arg z_1 - \arg z_2$.
- **Polar form:** $z = r(\cos\theta + i \sin\theta) = r \text{cis}\theta$, where $r = |z|$ and $\theta = \arg z$.
- **Exponential (Euler) form:** $e^{i\theta} = \cos\theta + i \sin\theta$, so $z = r e^{i\theta}$. Also $e^{i\pi} + 1 = 0$.
- **De Moivre's theorem:** $(\cos\theta + i \sin\theta)^n = \cos n\theta + i \sin n\theta$. Also $(\cos\theta - i \sin\theta)^n = \cos n\theta - i \sin n\theta$ and $(\cos\theta + i \sin\theta)^{-1} = \cos\theta - i \sin\theta$.
- **Useful substitution:** If $z = \cos\theta + i \sin\theta$, then $z^n + 1/z^n = 2\cos n\theta$ and $z^n - 1/z^n = 2i \sin n\theta$. Used for expanding powers of $\sin\theta$ and $\cos\theta$.
- **n th roots:** If $z = r \text{cis}\theta$, then $z^{1/n} = r^{1/n} [\cos((\theta + 2k\pi)/n) + i \sin((\theta + 2k\pi)/n)]$, $k = 0, 1, 2, \dots, n-1$. The n roots are equally spaced (angle $2\pi/n$) on a circle of radius $r^{1/n}$.
- **Cube roots of unity:** $1, \omega, \omega^2$, where $\omega = (-1 + i\sqrt{3})/2$ and $\omega^2 = (-1 - i\sqrt{3})/2$. Properties: $\omega^3 = 1$; $1 + \omega + \omega^2 = 0$; ω^2 is the conjugate of ω ; $1 + \omega = -\omega^2$; $1 + \omega^2 = -\omega$; $\omega^{3k} = 1$, $\omega^{3k+1} = \omega$, $\omega^{3k+2} = \omega^2$.
- **Locus:** $|z - z_0| = r$ is a circle with centre z_0 and radius r . $|z - z_1| = |z - z_2|$ is the perpendicular bisector of the line joining z_1 and z_2 .

Section A: Short Answer Questions (2 Marks)

A1. Evaluate (i) i^{25} (ii) i^{102} (iii) i^{-15} . [2 Marks]

Solution:

1. Rule: i^n depends only on the remainder when n is divided by 4 ($i^0 = 1$, $i^1 = i$, $i^2 = -1$, $i^3 = -i$).
2. (i) $25 = 4 \times 6 + 1$, so $i^{25} = i^1 = i$.
1. (ii) $102 = 4 \times 25 + 2$, so $i^{102} = i^2 = -1$.
2. (iii) $i^{-15} = 1/i^{15}$. Since $15 = 4 \times 3 + 3$, $i^{15} = i^3 = -i$. So $1/(-i) = -1/i = -(-i) = i$.

Answer: (i) i (ii) -1 (iii) i

A2. Find the conjugate of $(2 + 3i)/(1 - i)$. [2 Marks]

Solution:

1. Multiply numerator and denominator by the conjugate of the denominator, $(1 + i)$.
2. Numerator: $(2 + 3i)(1 + i) = 2 + 2i + 3i + 3i^2 = 2 + 5i - 3 = -1 + 5i$.
3. Denominator: $(1 - i)(1 + i) = 1 + 1 = 2$.
4. So $z = (-1 + 5i)/2 = -1/2 + (5/2)i$.
5. The conjugate is obtained by changing the sign of the imaginary part.

Answer: Conjugate = $-1/2 - (5/2)i$

A3. Find the modulus and principal argument of $z = -1 + i$. [2 Marks]

Solution:

1. Here $a = -1$, $b = 1$. $|z| = \sqrt{1 + 1} = \sqrt{2}$.
2. $\alpha = \tan^{-1}|b/a| = \tan^{-1}(1) = \pi/4$.
3. The point $(-1, 1)$ lies in the second quadrant, so $\theta = \pi - \alpha = \pi - \pi/4 = 3\pi/4$.

Answer: $|z| = \sqrt{2}$, $\arg z = 3\pi/4$

A4. Express $z = 1 + i\sqrt{3}$ in polar form. [2 Marks]

Solution:

1. $r = |z| = \sqrt{1 + 3} = 2$.
2. $\alpha = \tan^{-1}(\sqrt{3}/1) = \pi/3$. The point $(1, \sqrt{3})$ is in the first quadrant, so $\theta = \pi/3$.
3. Substitute in $z = r(\cos\theta + i \sin\theta)$.

Answer: $z = 2(\cos \pi/3 + i \sin \pi/3) = 2e^{i\pi/3}$

A5. Find the real and imaginary parts of $1/(3 - 4i)$. [2 Marks]

Solution:

1. Multiply numerator and denominator by $(3 + 4i)$.
2. $1/(3 - 4i) = (3 + 4i)/[(3 - 4i)(3 + 4i)] = (3 + 4i)/(9 + 16) = (3 + 4i)/25$.
3. Write in the form $a + ib$: $3/25 + i(4/25)$.

Answer: $\text{Re} = 3/25$, $\text{Im} = 4/25$

A6. State De Moivre's theorem and simplify $(\cos\theta + i \sin\theta)^4 (\cos\theta - i \sin\theta)^3$. [2 Marks]

Solution:

1. Statement: for any integer n , $(\cos\theta + i \sin\theta)^n = \cos n\theta + i \sin n\theta$.
2. Since $(\cos\theta - i \sin\theta) = (\cos\theta + i \sin\theta)^{-1}$, we get $(\cos\theta - i \sin\theta)^3 = (\cos\theta + i \sin\theta)^{-3}$.
3. Expression $= (\cos\theta + i \sin\theta)^4 \times (\cos\theta + i \sin\theta)^{-3} = (\cos\theta + i \sin\theta)^1$.

Answer: $\cos\theta + i \sin\theta$

A7. If ω is a complex cube root of unity, prove that $1 + \omega + \omega^2 = 0$. Hence find $\omega^{50} + \omega^{100}$. [2 Marks]

Solution:

1. Let $z^3 = 1$. Then $z^3 - 1 = 0$, i.e. $(z - 1)(z^2 + z + 1) = 0$.
2. For the complex roots $\omega \neq 1$, so $\omega^2 + \omega + 1 = 0$.
3. $50 = 3 \times 16 + 2$, so $\omega^{50} = \omega^2$. $100 = 3 \times 33 + 1$, so $\omega^{100} = \omega$.
4. $\omega^{50} + \omega^{100} = \omega^2 + \omega = -1$ (using $1 + \omega + \omega^2 = 0$).

Answer: $1 + \omega + \omega^2 = 0$ (proved); $\omega^{50} + \omega^{100} = -1$

A8. Evaluate $[(1 + i)/(1 - i)]^{25}$. [2 Marks]

Solution:

1. $(1 + i)/(1 - i) = (1 + i)^2 / [(1 - i)(1 + i)] = (1 + 2i + i^2)/2 = 2i/2 = i$.
2. So the expression $= i^{25}$.
3. $25 = 4 \times 6 + 1$, so $i^{25} = i$.

Answer: i

A9. Express $1 - i$ in exponential form. [2 Marks]

Solution:

1. $r = \sqrt{1^2 + 1^2} = \sqrt{2}$.
2. $\alpha = \tan^{-1}(1) = \pi/4$. The point $(1, -1)$ is in the fourth quadrant, so $\theta = -\pi/4$.
3. Use $z = r e^{i\theta}$.

Answer: $1 - i = \sqrt{2} e^{-i\pi/4}$

A10. Using Euler's formula, show that $e^{i\pi} + 1 = 0$. [2 Marks]

Solution:

1. Euler's formula: $e^{i\theta} = \cos\theta + i \sin\theta$.
2. Put $\theta = \pi$: $e^{i\pi} = \cos\pi + i \sin\pi = -1 + i(0) = -1$.
3. Hence $e^{i\pi} + 1 = -1 + 1 = 0$.

Answer: $e^{i\pi} + 1 = 0$ (proved)

Section B: Medium Answer Questions (4 Marks)

B1. Express $(2 + 3i)/(4 + 5i)$ in the form $a + ib$. [4 Marks]

Solution:

1. Multiply numerator and denominator by the conjugate of the denominator, $(4 - 5i)$.
2. Numerator: $(2 + 3i)(4 - 5i) = 8 - 10i + 12i - 15i^2 = 8 + 2i + 15 = 23 + 2i$.
3. Denominator: $(4 + 5i)(4 - 5i) = 16 + 25 = 41$.
4. Therefore the number $= (23 + 2i)/41 = 23/41 + i(2/41)$.

Answer: $23/41 + (2/41)i$, i.e. $\text{Re} = 23/41$ and $\text{Im} = 2/41$

B2. Find the modulus and argument of $z = (1 + 2i)/(1 - 3i)$. [4 Marks]

Solution:

1. Rationalise: $z = (1 + 2i)(1 + 3i)/[(1 - 3i)(1 + 3i)] = (1 + 3i + 2i + 6i^2)/(1 + 9) = (-5 + 5i)/10 = -1/2 + i/2$.
2. Modulus: $|z| = \sqrt{(1/4 + 1/4)} = \sqrt{(1/2)} = 1/\sqrt{2}$.
3. Argument: $\alpha = \tan^{-1}[(1/2)/(-1/2)] = \tan^{-1}(1) = \pi/4$. The point $(-1/2, 1/2)$ is in the second quadrant, so $\theta = \pi - \pi/4 = 3\pi/4$.
4. Check: $|1 + 2i|/|1 - 3i| = \sqrt{5}/\sqrt{10} = 1/\sqrt{2}$, which matches.

Answer: $|z| = 1/\sqrt{2}$, $\arg z = 3\pi/4$

B3. Find the real numbers x and y if $(x + iy)(2 - 3i) = 4 + i$. [4 Marks]

Solution:

1. $x + iy = (4 + i)/(2 - 3i)$.
2. Multiply numerator and denominator by $(2 + 3i)$: numerator $(4 + i)(2 + 3i) = 8 + 12i + 2i + 3i^2 = 5 + 14i$; denominator $= 4 + 9 = 13$.
3. So $x + iy = 5/13 + i(14/13)$.
4. Equate real parts and imaginary parts on both sides.

Answer: $x = 5/13$, $y = 14/13$

B4. Find the real values of x and y if $[(1 + i)x - 2i]/(3 + i) + [(2 - 3i)y + i]/(3 - i) = i$. [4 Marks]

Solution:

1. First term: $(1 + i)x - 2i = x + i(x - 2)$. Multiply by $(3 - i)/(3 - i)$: $[x + i(x - 2)](3 - i) = (4x - 2) + i(2x - 6)$. So first term $= [(4x - 2) + i(2x - 6)]/10$.
2. Second term: $(2 - 3i)y + i = 2y + i(1 - 3y)$. Multiply by $(3 + i)/(3 + i)$: $[2y + i(1 - 3y)](3 + i) = (9y - 1) + i(3 - 7y)$. So second term $= [(9y - 1) + i(3 - 7y)]/10$.
3. Add and multiply both sides by 10: $(4x + 9y - 3) + i(2x - 7y - 3) = 10i$.
4. Equate real parts: $4x + 9y = 3$... (1). Equate imaginary parts: $2x - 7y - 3 = 10$, i.e. $2x - 7y = 13$... (2).
5. Multiply (2) by 2: $4x - 14y = 26$. Subtract from (1): $23y = -23$, so $y = -1$.

6. Put $y = -1$ in (2): $2x + 7 = 13$, so $x = 3$.

Answer: $x = 3$, $y = -1$

Note: Check: for $x = 3$, $y = -1$ the first term equals 1 and the second equals $-1 + i$, and their sum is i .

B5. Find the square root of $3 + 4i$. [4 Marks]

Solution:

1. Let $\sqrt{3 + 4i} = x + iy$. Squaring: $x^2 - y^2 + 2ixy = 3 + 4i$.
2. Equate parts: $x^2 - y^2 = 3$...(1) and $2xy = 4$, i.e. $xy = 2$...(2).
3. Take modulus on both sides of $(x + iy)^2 = 3 + 4i$: $x^2 + y^2 = |3 + 4i| = \sqrt{9 + 16} = 5$...(3).
4. Add (1) and (3): $2x^2 = 8$, so $x^2 = 4$ and $x = \pm 2$. Subtract (1) from (3): $2y^2 = 2$, so $y^2 = 1$ and $y = \pm 1$.
5. From (2), $xy = 2$ is positive, so x and y have the same sign.

Answer: $\sqrt{3 + 4i} = \pm(2 + i)$, i.e. $2 + i$ or $-2 - i$

B6. Express $z = -1 - i\sqrt{3}$ in polar form and exponential form. [4 Marks]

Solution:

1. $r = \sqrt{1 + 3} = 2$.
2. $\alpha = \tan^{-1}(|\sqrt{3}|/1) = \pi/3$. The point $(-1, -\sqrt{3})$ is in the third quadrant, so $\theta = -(\pi - \pi/3) = -2\pi/3$.
3. Polar form: $z = 2[\cos(-2\pi/3) + i \sin(-2\pi/3)] = 2[\cos(2\pi/3) - i \sin(2\pi/3)]$.
4. Check: $2(-1/2 - i\sqrt{3}/2) = -1 - i\sqrt{3}$.

Answer: $z = 2[\cos(2\pi/3) - i \sin(2\pi/3)] = 2e^{-i2\pi/3}$

B7. If $z_1 = 1 + i$ and $z_2 = \sqrt{3} - i$, find $|z_1 z_2|$, $|z_1/z_2|$, $\arg(z_1 z_2)$ and $\arg(z_1/z_2)$. [4 Marks]

Solution:

1. $|z_1| = \sqrt{2}$, $|z_2| = \sqrt{3 + 1} = 2$.
2. $\arg z_1 = \pi/4$ (first quadrant). For z_2 : $\alpha = \tan^{-1}(1/\sqrt{3}) = \pi/6$, fourth quadrant, so $\arg z_2 = -\pi/6$.
3. $|z_1 z_2| = |z_1||z_2| = 2\sqrt{2}$. $|z_1/z_2| = \sqrt{2}/2 = 1/\sqrt{2}$.
4. $\arg(z_1 z_2) = \pi/4 + (-\pi/6) = \pi/12$. $\arg(z_1/z_2) = \pi/4 - (-\pi/6) = 5\pi/12$.

Answer: $2\sqrt{2}$, $1/\sqrt{2}$, $\pi/12$, $5\pi/12$

B8. Find the locus of the point $z = x + iy$ if $|z - 1| = |z + i|$. [4 Marks]

Solution:

1. Write $z = x + iy$. Then $|z - 1| = |(x - 1) + iy|$ and $|z + i| = |x + i(y + 1)|$.
2. Given condition (squaring both sides): $(x - 1)^2 + y^2 = x^2 + (y + 1)^2$.
3. Expand: $x^2 - 2x + 1 + y^2 = x^2 + y^2 + 2y + 1$.
4. Simplify: $-2x = 2y$, i.e. $x + y = 0$.

Answer: Locus is the straight line $x + y = 0$ ($y = -x$), the perpendicular bisector of the points 1 and $-i$.

B9. Evaluate using polar form (i) $(1 + i)^{10}$ (ii) $(\sqrt{3} + i)^9$. [4 Marks]

Solution:

1. (i) $1 + i = \sqrt{2} (\cos \pi/4 + i \sin \pi/4)$. By De Moivre: $(1 + i)^{10} = (\sqrt{2})^{10} [\cos(10\pi/4) + i \sin(10\pi/4)] = 32[\cos(5\pi/2) + i \sin(5\pi/2)] = 32(0 + i) = 32i$.
2. (ii) $\sqrt{3} + i$: $r = 2$, $\theta = \pi/6$, so $\sqrt{3} + i = 2(\cos \pi/6 + i \sin \pi/6)$.
3. $(\sqrt{3} + i)^9 = 2^9 [\cos(9\pi/6) + i \sin(9\pi/6)] = 512[\cos(3\pi/2) + i \sin(3\pi/2)] = 512(0 - i) = -512i$.

Answer: (i) 32i (ii) -512i

B10. Simplify $[(\cos 2\theta + i \sin 2\theta)^3 (\cos 3\theta - i \sin 3\theta)^{-2}] / [(\cos 4\theta + i \sin 4\theta)^5 (\cos 5\theta - i \sin 5\theta)^{-4}]$. [4 Marks]

Solution:

1. Write every factor in the form $\cos n\theta + i \sin n\theta$. Note $\cos n\theta - i \sin n\theta = \cos(-n\theta) + i \sin(-n\theta)$.
2. Numerator: $(\cos 2\theta + i \sin 2\theta)^3 = \cos 6\theta + i \sin 6\theta$. $(\cos 3\theta - i \sin 3\theta)^{-2} = [\cos(-3\theta) + i \sin(-3\theta)]^{-2} = \cos 6\theta + i \sin 6\theta$. So numerator $= (\cos 6\theta + i \sin 6\theta)^2 = \cos 12\theta + i \sin 12\theta$.
3. Denominator: $(\cos 4\theta + i \sin 4\theta)^5 = \cos 20\theta + i \sin 20\theta$. $(\cos 5\theta - i \sin 5\theta)^{-4} = \cos 20\theta + i \sin 20\theta$. So denominator $= \cos 40\theta + i \sin 40\theta$.
4. Ratio: $(\cos 12\theta + i \sin 12\theta) / (\cos 40\theta + i \sin 40\theta) = \cos(12\theta - 40\theta) + i \sin(12\theta - 40\theta) = \cos(-28\theta) + i \sin(-28\theta)$.

Answer: $\cos 28\theta - i \sin 28\theta$

B11. Find the cube roots of unity and show that their sum is zero and their product is 1. [4 Marks]

Solution:

1. Let $x = 1^{1/3}$, so $x^3 = 1$ or $x^3 - 1 = 0$, i.e. $(x - 1)(x^2 + x + 1) = 0$.
2. $x = 1$, or $x = [-1 \pm \sqrt{1 - 4}]/2 = (-1 \pm i\sqrt{3})/2$.
3. Let $\omega = (-1 + i\sqrt{3})/2$. Then $\omega^2 = (1 - 2i\sqrt{3} - 3)/4 = (-1 - i\sqrt{3})/2$. So the roots are 1, ω , ω^2 .
4. Sum: $1 + \omega + \omega^2 = 1 + [(-1 + i\sqrt{3}) + (-1 - i\sqrt{3})]/2 = 1 - 1 = 0$.
5. Product: $1 \cdot \omega \cdot \omega^2 = \omega^3 = 1$.

Answer: Roots: 1, $(-1 + i\sqrt{3})/2$, $(-1 - i\sqrt{3})/2$; sum = 0; product = 1

B12. If ω is a cube root of unity, prove that (i) $(1 - \omega + \omega^2)(1 + \omega - \omega^2) = 4$ (ii) $(1 - \omega + \omega^2)^5 + (1 + \omega - \omega^2)^5 = 32$. [4 Marks]

Solution:

1. Use $1 + \omega + \omega^2 = 0$, so $1 + \omega^2 = -\omega$ and $1 + \omega = -\omega^2$.
2. $1 - \omega + \omega^2 = (1 + \omega^2) - \omega = -\omega - \omega = -2\omega$.
3. $1 + \omega - \omega^2 = (1 + \omega) - \omega^2 = -\omega^2 - \omega^2 = -2\omega^2$.
4. (i) Product $= (-2\omega)(-2\omega^2) = 4\omega^3 = 4$.

5. (ii) $(-2\omega)^5 + (-2\omega^2)^5 = -32\omega^5 - 32\omega^{10} = -32\omega^2 - 32\omega$ (as $\omega^5 = \omega^2$, $\omega^{10} = \omega$) $= -32(\omega + \omega^2) = -32(-1) = 32$.

Answer: (i) 4 (proved) (ii) 32 (proved)

B13. Prove that $(1 + i)^n + (1 - i)^n = 2^{n/2+1} \cos(n\pi/4)$. [4 Marks]

Solution:

1. $1 + i = \sqrt{2} (\cos \pi/4 + i \sin \pi/4)$ and $1 - i = \sqrt{2} (\cos \pi/4 - i \sin \pi/4)$.
2. By De Moivre: $(1 + i)^n = 2^{n/2} (\cos n\pi/4 + i \sin n\pi/4)$ and $(1 - i)^n = 2^{n/2} (\cos n\pi/4 - i \sin n\pi/4)$.
3. Add: $(1 + i)^n + (1 - i)^n = 2^{n/2} \times 2 \cos(n\pi/4) = 2^{n/2+1} \cos(n\pi/4)$.

Answer: Hence proved.

B14. Find the cube roots of $1 + i$. [4 Marks]

Solution:

1. $1 + i$: $r = \sqrt{2}$, $\theta = \pi/4$. So $1 + i = \sqrt{2} [\cos(\pi/4 + 2k\pi) + i \sin(\pi/4 + 2k\pi)]$.
2. $(1 + i)^{1/3} = (\sqrt{2})^{1/3} [\cos((\pi/4 + 2k\pi)/3) + i \sin((\pi/4 + 2k\pi)/3)] = 2^{1/6} [\cos(\pi/12 + 2k\pi/3) + i \sin(\pi/12 + 2k\pi/3)]$, $k = 0, 1, 2$.
3. $k = 0$: angle $\pi/12$. $k = 1$: angle $\pi/12 + 2\pi/3 = 9\pi/12 = 3\pi/4$. $k = 2$: angle $\pi/12 + 4\pi/3 = 17\pi/12$.

Answer: $2^{1/6}(\cos \pi/12 + i \sin \pi/12)$, $2^{1/6}(\cos 3\pi/4 + i \sin 3\pi/4)$, $2^{1/6}(\cos 17\pi/12 + i \sin 17\pi/12)$

Section C: Long Answer Questions (6 Marks)

C1. Find all values of $(-8 + 8\sqrt{3}i)^{1/4}$ and express them in the form $a + ib$. [6 Marks]

Solution:

1. $r = \sqrt{(64 + 192)} = \sqrt{256} = 16$.
2. $\alpha = \tan^{-1}|8\sqrt{3}/(-8)| = \tan^{-1}(\sqrt{3}) = \pi/3$. The point $(-8, 8\sqrt{3})$ is in the second quadrant, so $\theta = \pi - \pi/3 = 2\pi/3$.
3. $z = 16[\cos(2\pi/3 + 2k\pi) + i \sin(2\pi/3 + 2k\pi)]$.
4. $z^{1/4} = 16^{1/4}[\cos((2\pi/3 + 2k\pi)/4) + i \sin((2\pi/3 + 2k\pi)/4)] = 2[\cos(\pi/6 + k\pi/2) + i \sin(\pi/6 + k\pi/2)]$, $k = 0, 1, 2, 3$.
5. $k = 0$: $2(\cos \pi/6 + i \sin \pi/6) = \sqrt{3} + i$.
6. $k = 1$: $2(\cos 2\pi/3 + i \sin 2\pi/3) = -1 + i\sqrt{3}$.
7. $k = 2$: $2(\cos 7\pi/6 + i \sin 7\pi/6) = -\sqrt{3} - i$.
8. $k = 3$: $2(\cos 5\pi/3 + i \sin 5\pi/3) = 1 - i\sqrt{3}$.

Answer: $\sqrt{3} + i$, $-1 + i\sqrt{3}$, $-\sqrt{3} - i$, $1 - i\sqrt{3}$

Note: The four roots lie on a circle of radius 2, spaced 90° apart.

C2. Using De Moivre's theorem, solve $x^4 - x^3 + x^2 - x + 1 = 0$. [6 Marks]

Solution:

1. Multiply the equation by $(x + 1)$: $(x + 1)(x^4 - x^3 + x^2 - x + 1) = x^5 + 1$.
2. So the roots of the given equation are the roots of $x^5 + 1 = 0$ except $x = -1$.
3. $x^5 = -1 = \cos\pi + i \sin\pi = \cos(\pi + 2k\pi) + i \sin(\pi + 2k\pi)$.
4. $x = \cos((2k + 1)\pi/5) + i \sin((2k + 1)\pi/5)$, $k = 0, 1, 2, 3, 4$.
5. $k = 0$: $\pi/5$. $k = 1$: $3\pi/5$. $k = 2$: π (this gives $x = -1$, which is rejected). $k = 3$: $7\pi/5$. $k = 4$: $9\pi/5$.

Answer: $x = \cos \pi/5 + i \sin \pi/5$, $\cos 3\pi/5 + i \sin 3\pi/5$, $\cos 7\pi/5 + i \sin 7\pi/5$, $\cos 9\pi/5 + i \sin 9\pi/5$

C3. Find all the sixth roots of unity and state where they lie on the Argand plane. [6 Marks]

Solution:

1. Let $x = 1^{1/6}$, so $x^6 = 1 = \cos(2k\pi) + i \sin(2k\pi)$.
2. $x = \cos(2k\pi/6) + i \sin(2k\pi/6) = \cos(k\pi/3) + i \sin(k\pi/3)$, $k = 0, 1, 2, 3, 4, 5$.
3. $k = 0$: $x = 1$.
4. $k = 1$: $\cos \pi/3 + i \sin \pi/3 = 1/2 + i\sqrt{3}/2$.
5. $k = 2$: $\cos 2\pi/3 + i \sin 2\pi/3 = -1/2 + i\sqrt{3}/2$.
6. $k = 3$: $\cos \pi + i \sin \pi = -1$.
7. $k = 4$: $\cos 4\pi/3 + i \sin 4\pi/3 = -1/2 - i\sqrt{3}/2$.
8. $k = 5$: $\cos 5\pi/3 + i \sin 5\pi/3 = 1/2 - i\sqrt{3}/2$.

Answer: $1, 1/2 \pm i\sqrt{3}/2, -1/2 \pm i\sqrt{3}/2, -1$

Note: All six roots lie on the unit circle and form the vertices of a regular hexagon. Their sum is zero.

C4. Using De Moivre's theorem, prove that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ and $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$. [6 Marks]

Solution:

1. Let $c = \cos\theta$ and $s = \sin\theta$. By De Moivre, $\cos 3\theta + i \sin 3\theta = (c + is)^3$.
2. Expand by the binomial theorem: $(c + is)^3 = c^3 + 3c^2(is) + 3c(is)^2 + (is)^3 = c^3 + 3ic^2s - 3cs^2 - is^3$.
3. Separate real and imaginary parts: $\cos 3\theta = c^3 - 3cs^2$ and $\sin 3\theta = 3c^2s - s^3$.
4. $\cos 3\theta$: put $s^2 = 1 - c^2$: $c^3 - 3c(1 - c^2) = c^3 - 3c + 3c^3 = 4c^3 - 3c$.
5. $\sin 3\theta$: put $c^2 = 1 - s^2$: $3(1 - s^2)s - s^3 = 3s - 3s^3 - s^3 = 3s - 4s^3$.

Answer: $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$, $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$ (proved)

C5. Expand $\cos 4\theta$, $\sin 4\theta$ and $\tan 4\theta$ using De Moivre's theorem. [6 Marks]

Solution:

1. Let $c = \cos\theta$, $s = \sin\theta$. $\cos 4\theta + i \sin 4\theta = (c + is)^4$.

- $(c + is)^4 = c^4 + 4c^3(is) + 6c^2(is)^2 + 4c(is)^3 + (is)^4 = c^4 + 4ic^3s - 6c^2s^2 - 4ics^3 + s^4$.
- Real part: $\cos 4\theta = c^4 - 6c^2s^2 + s^4$. Put $s^2 = 1 - c^2$: $c^4 - 6c^2(1 - c^2) + (1 - c^2)^2 = c^4 - 6c^2 + 6c^4 + 1 - 2c^2 + c^4 = 8c^4 - 8c^2 + 1$.
- Imaginary part: $\sin 4\theta = 4c^3s - 4cs^3 = 4cs(c^2 - s^2)$. Put $s^2 = 1 - c^2$: $= 4c^3s - 4cs(1 - c^2) = 8c^3s - 4cs = \sin \theta (8\cos^3 \theta - 4\cos \theta)$.
- $\tan 4\theta = \sin 4\theta / \cos 4\theta = (4c^3s - 4cs^3) / (c^4 - 6c^2s^2 + s^4)$. Divide numerator and denominator by c^4 , and put $t = \tan \theta$: $\tan 4\theta = (4t - 4t^3) / (1 - 6t^2 + t^4)$.

Answer: $\cos 4\theta = 8\cos^4 \theta - 8\cos^2 \theta + 1$; $\sin 4\theta = 8\cos^3 \theta \sin \theta - 4\cos \theta \sin \theta$; $\tan 4\theta = (4\tan \theta - 4\tan^3 \theta) / (1 - 6\tan^2 \theta + \tan^4 \theta)$

C6. Expand $\sin^5 \theta$ in a series of sines of multiples of θ . [6 Marks]

Solution:

- Let $z = \cos \theta + i \sin \theta$. Then $z - 1/z = 2i \sin \theta$, and $z^n - 1/z^n = 2i \sin n\theta$.
- $(2i \sin \theta)^5 = (z - 1/z)^5$. LHS = $32i^5 \sin^5 \theta = 32i \sin^5 \theta$.
- Expand RHS: $(z - 1/z)^5 = z^5 - 5z^3 + 10z - 10/z + 5/z^3 - 1/z^5$.
- Group terms: $= (z^5 - 1/z^5) - 5(z^3 - 1/z^3) + 10(z - 1/z) = 2i \sin 5\theta - 10i \sin 3\theta + 20i \sin \theta$.
- So $32i \sin^5 \theta = 2i(\sin 5\theta - 5 \sin 3\theta + 10 \sin \theta)$. Divide by $32i$.

Answer: $\sin^5 \theta = (1/16)(\sin 5\theta - 5 \sin 3\theta + 10 \sin \theta)$

Note: Check with $\theta = \pi/2$: LHS = 1 and RHS = $(1/16)(1 + 5 + 10) = 1$.

C7. Expand $\cos^4 \theta$ in a series of cosines of multiples of θ . [6 Marks]

Solution:

- Let $z = \cos \theta + i \sin \theta$. Then $z + 1/z = 2\cos \theta$, and $z^n + 1/z^n = 2\cos n\theta$.
- $(2\cos \theta)^4 = (z + 1/z)^4$. LHS = $16\cos^4 \theta$.
- Expand RHS: $z^4 + 4z^2 + 6 + 4/z^2 + 1/z^4 = (z^4 + 1/z^4) + 4(z^2 + 1/z^2) + 6$.
- $= 2\cos 4\theta + 8\cos 2\theta + 6$.
- So $16\cos^4 \theta = 2\cos 4\theta + 8\cos 2\theta + 6$. Divide by 16.

Answer: $\cos^4 \theta = (1/8)(\cos 4\theta + 4\cos 2\theta + 3)$

Note: By the same method, $\cos^3 \theta = (1/4)(\cos 3\theta + 3\cos \theta)$. Check with $\theta = 0$: LHS = 1 and RHS = $(1/8)(1 + 4 + 3) = 1$.

C8. Prove that $(1 + \cos \theta + i \sin \theta)^n + (1 + \cos \theta - i \sin \theta)^n = 2^{n+1} \cos^n(\theta/2) \cos(n\theta/2)$. [6 Marks]

Solution:

- Use $1 + \cos \theta = 2\cos^2(\theta/2)$ and $\sin \theta = 2\sin(\theta/2)\cos(\theta/2)$.
- $1 + \cos \theta + i \sin \theta = 2\cos^2(\theta/2) + 2i \sin(\theta/2)\cos(\theta/2) = 2\cos(\theta/2)[\cos(\theta/2) + i \sin(\theta/2)]$.
- Similarly, $1 + \cos \theta - i \sin \theta = 2\cos(\theta/2)[\cos(\theta/2) - i \sin(\theta/2)]$.
- Raise both to the power n and apply De Moivre: $(1 + \cos \theta + i \sin \theta)^n = 2^n \cos^n(\theta/2)[\cos(n\theta/2) + i \sin(n\theta/2)]$ and $(1 + \cos \theta - i \sin \theta)^n = 2^n \cos^n(\theta/2)[\cos(n\theta/2) - i \sin(n\theta/2)]$.
- Add: LHS = $2^n \cos^n(\theta/2) \times 2\cos(n\theta/2) = 2^{n+1} \cos^n(\theta/2) \cos(n\theta/2) = \text{RHS}$.

Answer: Hence proved.

C9. Prove that $[(1 + \sin\theta + i \cos\theta)/(1 + \sin\theta - i \cos\theta)]^n = \cos(n\pi/2 - n\theta) + i \sin(n\pi/2 - n\theta)$. [6 Marks]

Solution:

1. Put $\varphi = \pi/2 - \theta$. Then $\sin\theta = \cos\varphi$ and $\cos\theta = \sin\varphi$.
2. Numerator: $1 + \cos\varphi + i \sin\varphi = 2\cos^2(\varphi/2) + 2i \sin(\varphi/2)\cos(\varphi/2) = 2\cos(\varphi/2)[\cos(\varphi/2) + i \sin(\varphi/2)]$.
3. Denominator: $1 + \cos\varphi - i \sin\varphi = 2\cos(\varphi/2)[\cos(\varphi/2) - i \sin(\varphi/2)]$.
4. Ratio = $[\cos(\varphi/2) + i \sin(\varphi/2)]/[\cos(\varphi/2) - i \sin(\varphi/2)] = [\cos(\varphi/2) + i \sin(\varphi/2)]^2 = \cos\varphi + i \sin\varphi$.
5. Raise to the power n (De Moivre): $(\cos\varphi + i \sin\varphi)^n = \cos n\varphi + i \sin n\varphi$.
6. Substitute $\varphi = \pi/2 - \theta$: $\cos(n\pi/2 - n\theta) + i \sin(n\pi/2 - n\theta)$.

Answer: Hence proved.

Section D: Practice Questions with Answers

Solve these on your own first and then check the answers.

1. Find i^{4n+3} . **Ans: $-i$**
2. Find the modulus of $3 - 4i$. **Ans: 5**
3. Find the principal argument of $-i$. **Ans: $-\pi/2$**
4. Find the conjugate of $(1 + i)^2$. **Ans: $-2i$ (since $(1 + i)^2 = 2i$)**
5. Find the real and imaginary parts of $(2 + i)^2$. **Ans: Re = 3, Im = 4**
6. Find z if $z + 2\bar{z} = 3 + i$. **Ans: $z = 1 - i$ (put $z = x + iy$, get $3x = 3$ and $-y = 1$)**
7. Find $|(1 + i)^8|$. **Ans: 16**
8. Simplify $(\cos\pi/3 + i \sin\pi/3)^6$. **Ans: 1**
9. Find the value of $(1 + \omega^2)^3$. **Ans: -1 (since $1 + \omega^2 = -\omega$ and $\omega^3 = 1$)**
10. Find the product of all cube roots of unity. **Ans: 1**
11. How many distinct n th roots does a non-zero complex number have? **Ans: Exactly n**
12. Write $1 + i$ in exponential form. **Ans: $\sqrt{2} e^{i\pi/4}$**

Exam Tips

- Always locate the quadrant before writing the argument. A wrong quadrant is the most common mistake.
- For division of complex numbers, rationalise by multiplying with the conjugate of the denominator, and write the final answer as $a + ib$.
- For n th roots, write the number in polar form with $2k\pi$ added to the angle, then give $k = 0$ to $n - 1$. State every root.
- For expansion of $\sin^n\theta$ and $\cos^n\theta$, use $z \pm 1/z$ with the binomial theorem, and group the terms in pairs.

- Use $1 + \omega + \omega^2 = 0$ and $\omega^3 = 1$ to reduce every cube-root-of-unity problem.
- Write the given formula or theorem first, then the steps, then a clearly marked final answer to secure full marks.

Poly Notes Hub