MATHEMATICS 1

Chapter: Logarithm

Broad Questions and Answers (Descriptive Type)

For Engineering and Polytechnic Students | Exam-Oriented

Note: In this document, "log" means $\log_{10}$ (base 10) unless another base is written, and "ln" means $\log_e$. Part 1 contains theory questions with detailed explanations, and Part 2 contains long problems with complete step-by-step solutions.

Part 1: Theory Questions (Descriptive Answers)

Q1. What is a logarithm? Explain the relation between the exponential form and the logarithmic form. State the conditions on the base and the number, with reasons, and give examples. **[6 Marks]**

Answer:

Definition. If a positive number a ($a \neq 1$) is raised to a power x to give a number N, that is $a^x = N$, then x is called the logarithm of N to the base a and is written $x = \log_a N$. In simple words, a logarithm is the power (index) to which the base must be raised to get the given number.

Relation between the two forms. The exponential form $a^x = N$ and the logarithmic form $\log_a N = x$ carry exactly the same information. During conversion the base stays the base, the power becomes the value of the logarithm, and the result N becomes the number whose logarithm is taken.

Examples of conversion

- $2^5 = 32 \leftrightarrow \log_2 32 = 5$
- $10^3 = 1000 \leftrightarrow \log_{10} 1000 = 3$
- $5^{-2} = 1/25 \leftrightarrow \log_5 (1/25) = -2$
- $9^{1/2} = 3 \leftrightarrow \log_9 3 = 1/2$

Conditions on the base and the number

- The base must be positive ($a > 0$). For a negative base, powers such as $a^{1/2}$ are not real numbers, so the logarithm would be undefined for many values.
- The base must not be 1 ($a \neq 1$). Since $1^x = 1$ for every x, $\log_1 1$ would have infinitely many values, and $\log_1 N$ would not exist for any other N.
- The number must be positive ($N > 0$). A positive base raised to any real power always gives a positive result, so no power of a can be zero or negative. Hence $\log_a 0$ and the logarithm of a negative number do not exist in real numbers.

Two results that follow directly

- $\log_a 1 = 0$, because $a^0 = 1$.
- $\log_a a = 1$, because $a^1 = a$.

Conclusion. A logarithm turns multiplication into addition and powers into multiplication. This is why logarithms are so useful in calculation and in engineering scales.

Q2. State and prove the product law, quotient law and power law of logarithms. Verify each with a numerical example and mention the common mistakes. **[8 Marks]**

Answer:

Let M and N be positive numbers and $a > 0$, $a \neq 1$. Let $\log_a M = p$ and $\log_a N = q$, so that $M = a^p$ and $N = a^q$.

1. Product law: log_{a} (MN) = log_{a} M + log_{a} N

Proof: $MN = a^p \times a^q = a^{p+q}$ (law of indices). Changing to logarithmic form, $\log_a (MN) = p + q = \log_a M + \log_a N$.

Example: $\log_2 (8 \times 4) = \log_2 32 = 5$, and $\log_2 8 + \log_2 4 = 3 + 2 = 5$. Verified.

2. Quotient law: log_{a} (M/N) = log_{a} M − log_{a} N

Proof: $M/N = a^p / a^q = a^{p-q}$. Hence $\log_a (M/N) = p - q = \log_a M - \log_a N$.

Example: $\log_2 (32/8) = \log_2 4 = 2$, and $\log_2 32 - \log_2 8 = 5 - 3 = 2$. Verified.

3. Power law: log_{a} (M^{n}) = n log_{a} M

Proof: $M^n = (a^p)^n = a^{np}$. Hence $\log_a (M^n) = np = n \log_a M$.

Example: $\log_2 (8^2) = \log_2 64 = 6$, and $2 \times \log_2 8 = 2 \times 3 = 6$. Verified.

Special cases of the power law

- Reciprocal: $\log_a (1/M) = -\log_a M$.
- Root: $\log_a (M^{1/n}) = (1/n) \log_a M$.

Common mistakes

- $\log (A + B)$ is not equal to $\log A + \log B$.
- $\log A / \log B$ is not equal to $\log (A / B)$.
- $(\log A)^n$ is not equal to $n \log A$.
- All the laws are valid only when every number inside a logarithm is positive.

Q3. Derive the change of base formula. Deduce the reciprocal rule, the chain rule and the result for a base that is a power of a (base a^n). Use them to evaluate $\log_8 32$ and $\log_{25} 125$. **[8 Marks]**

Answer:

Derivation

1. Let $\log_a M = x$. In exponential form $a^x = M$.
2. Take logarithm to the base b on both sides: $x \log_b a = \log_b M$.
3. Therefore $x = \log_b M / \log_b a$, that is $\log_a M = \log_b M / \log_b a$. This is the change of base formula.

Deductions

- **Reciprocal rule:** put M = b. Then $\log_a b = \log_b b / \log_b a = 1 / \log_b a$. So $\log_a b \times \log_b a = 1$.
- **Chain rule:** change to base 10: $\log_a b \times \log_b c = (\log b / \log a) \times (\log c / \log b) = \log c / \log a = \log_a c$. In the same way $\log_a b \times \log_b c \times \log_c d = \log_a d$.
- **Base as a power:** $\log_{a^{\wedge}\{n}\}$ M is the log of M to the base a^n. Using the formula, it equals $\log_a M / \log_a a^n = (1/n) \log_a M$.

Examples

- $\log_8 32 = \log_2 32 / \log_2 8 = 5/3$.
- $\log_{25} 125 = \log_5 125 / \log_5 25 = 3/2$.
- $1/\log_2 10 + 1/\log_5 10 = \log_{10} 2 + \log_{10} 5 = \log_{10} 10 = 1$.

Use in practice. Calculators give only log (base 10) and ln (base e). Any other base is handled by this formula. For instance $\log_2 7 = \log 7 / \log 2 = 0.8451 / 0.3010 \approx 2.81$.

Q4. Explain common logarithm and natural logarithm. Derive the relation between them and state where each is used in engineering. **[6 Marks]**

Answer:

Common logarithm has base 10 and is written log or $\log_{10}$. **Natural logarithm** has base e ≈ 2.718 (an irrational number) and is written ln or $\log_e$.

Derivation of the relation

1. Let ln N = x. Then $e^x = N$.
2. Take common logarithm on both sides: x log e = log N, so x = log N / log e.
3. The value of $\log_{10} e$ is 0.4343, so ln N = log N / 0.4343 = 2.3026 × log N.

Hence ln N = 2.3026 × $\log_{10}$ N and $\log_{10}$ N = 0.4343 × ln N. For example ln 100 = 2.3026 × 2 = 4.6052.

Uses

- **Common logarithm:** numerical calculation, characteristic and mantissa work, and logarithmic scales such as the decibel scale.
- **Natural logarithm:** calculus (the derivative of ln x is 1/x), growth and decay problems, RC and RL circuit transients, radioactive decay and continuous compounding.

Q5. Explain characteristic and mantissa of a common logarithm with rules and examples. Show how they are used to find the number of digits in a number and the number of zeros after the decimal point. **[8 Marks]**

Answer:

Every common logarithm can be written as log N = characteristic + mantissa. The characteristic is the integer part and the mantissa is the decimal part, which is always positive and lies between 0 and 1.

Rules for the characteristic

- If N ≥ 1: characteristic = (number of digits before the decimal point) − 1.
- If 0 < N < 1: characteristic = −(number of zeros just after the decimal point + 1). It is negative and is written with a bar on top.

The mantissa depends only on the digits

Numbers with the same digits have the same mantissa. Only the characteristic changes with the position of the decimal point. Taking log 2 = 0.3010:

- log 2000 = 3.3010, log 200 = 2.3010, log 20 = 1.3010, log 2 = 0.3010
- log 0.2 = −1 + 0.3010, log 0.02 = −2 + 0.3010, log 0.002 = −3 + 0.3010

Writing a negative logarithm: the mantissa must stay positive. So −6.020 is written as −7 + 0.980 (characteristic −7, mantissa 0.980).

Application 1: number of digits

Number of digits in N = characteristic of log N + 1. Example: log 2^{100} = 100 × 0.3010 = 30.10. The characteristic is 30, so 2^{100} has 31 digits.

Application 2: zeros after the decimal point

If the characteristic is −n, the first non-zero digit appears at the n-th place after the decimal point, so the number of zeros after the point is n − 1. Example: for $(1/2)^{20}$, log = −20 × 0.3010 = −6.020 = −7 + 0.980. The characteristic is −7, so there are 7 − 1 = 6 zeros after the decimal point.

Q6. Discuss the properties of the logarithmic function $y = \log_a x$, its relation with the exponential function, and why logarithmic scales are used in engineering. **[6 Marks]**

Answer:

- **Domain and range:** the domain is $x > 0$ and the range is all real numbers.
- **Passes through (1, 0):** because $\log_a 1 = 0$ for every valid base.
- **Monotonic behaviour:** if $a > 1$ the function is increasing; if $0 < a < 1$ it is decreasing. Hence it is one-one, and $\log_a M = \log_a N$ implies $M = N$.
- **Sign (for a > 1):** $\log_a x$ is negative for $0 < x < 1$ and positive for $x > 1$.
- **Behaviour near zero (for a > 1):** as x approaches 0 from the right, $\log_a x$ decreases without limit. The y-axis is a vertical asymptote of the graph.
- **Inverse of the exponential function:** $y = \log_a x$ and $y = a^x$ are inverses of each other, so the graph of one is the mirror image of the other in the line $y = x$.

Sample values for log_{10} x

$x = 0.01, 0.1, 1, 10, 100$ give $y = −2, −1, 0, 1, 2$ respectively.

Why logarithmic scales are used

The logarithm grows very slowly. When x rises from 10 to 1000, $\log_{10} x$ rises only from 1 to 3. This lets engineers compress quantities that vary over a huge range (sound intensity, signal power, earthquake energy) into small, easy-to-handle numbers.

Q7. Explain the method of solving logarithmic and exponential equations. Give a worked example for each of the five common types. **[10 Marks]**

Answer:

Golden rule: before solving, write the condition that every argument of a logarithm is positive (and every base is positive and not 1). After solving, reject any root that breaks these conditions.

Type 1: a single logarithm on each side

Solve $\log_3 (2x + 1) = \log_3 (x + 5)$.

1. Conditions: $2x + 1 > 0$ and $x + 5 > 0$.
2. Equate the arguments (one-one property): $2x + 1 = x + 5$, so $x = 4$.
3. Check: $2(4) + 1 = 9 > 0$ and $4 + 5 = 9 > 0$. So $x = 4$ is valid.

Type 2: sum or difference of logarithms

Solve $\log (x + 1) + \log (x − 2) = 1$.

1. Condition: $x − 2 > 0$, so $x > 2$.
2. Combine: $\log [(x + 1)(x − 2)] = 1$, so $(x + 1)(x − 2) = 10^1 = 10$.
3. $x^2 − x − 2 = 10$, so $x^2 − x − 12 = 0$, i.e. $(x − 4)(x + 3) = 0$.
4. $x = 4$ or $x = −3$. Reject $x = −3$ because $x > 2$. So $x = 4$.

Type 3: logarithms with different bases

Solve $\log_2 x + \log_8 x = 4$.

1. Change to base 2: $\log_8 x = \log_2 x / \log_2 8 = (1/3) \log_2 x$.
2. Equation becomes $(1 + 1/3) \log_2 x = 4$, so $(4/3) \log_2 x = 4$.
3. Hence $\log_2 x = 3$ and $x = 2^3 = 8$.

Type 4: exponential equation (unknown in the power)

Solve $5^x = 30$, given $\log 3 = 0.4771$ and $\log 5 = 0.6990$.

1. Take log on both sides: x log 5 = log 30 = log 3 + log 10 = 0.4771 + 1 = 1.4771.
2. x = 1.4771 / 0.6990 = 2.113 (approximately).

Type 5: unknown base

Solve $\log_x (6 - x) = 2$.

1. Conditions: $x > 0$, $x \neq 1$ and $6 - x > 0$.
2. Exponential form: $x^2 = 6 - x$, so $x^2 + x - 6 = 0$, i.e. $(x + 3)(x - 2) = 0$.
3. x = −3 is rejected because a base must be positive. So x = 2.

Q8. Explain, with worked examples, any four applications of logarithms in engineering and science. **[8 Marks]**

Answer:

Logarithms compress very large ranges of values and turn multiplicative changes into additive ones. Four common uses are given below.

1. Decibel (dB) scale

For power ratio, dB = 10 log (P_2/P_1). For voltage ratio, dB = 20 log (V_2/V_1).

- An amplifier with voltage gain 100 has gain 20 log 100 = 20 × 2 = 40 dB.
- If the output power is half the input power, the change is 10 log (1/2) = −10 × 0.3010 = −3.01 dB. This is the well-known −3 dB (half-power) point.

2. Discharge of a capacitor (RC circuit)

During discharge $v = V e^{-t/RC}$. Find the time to fall to 5% of V.

1. $e^{-t/RC} = 0.05 = 1/20$.
2. Take ln: −t/RC = −ln 20, so t = RC ln 20 = RC (ln 2 + ln 10) = RC (0.6931 + 2.3026) ≈ 3 RC.

3. Radioactive decay (half-life)

$N = N_0 (1/2)^{t/T}$, where T is the half-life. Find the time for the sample to fall to 10% of its initial amount.

1. $(1/2)^{t/T} = 0.1$. Take log: (t/T) log (1/2) = log 0.1.
2. (t/T)(−0.3010) = −1, so t/T = 1/0.3010 = 3.32. Hence t ≈ 3.32 T.

4. Compound interest

$A = P (1 + r)^n$. Find the time for a sum to triple at 10% per year, given log 3 = 0.4771 and log 1.1 = 0.0414.

1. $3P = P (1.1)^n$, so $(1.1)^n = 3$.
2. n log 1.1 = log 3, so n = 0.4771 / 0.0414 ≈ 11.5 years.

Part 2: Long Problems (Detailed Solutions)

Q9. Prove that $\log_a b \times \log_b c \times \log_c d = \log_a d$. Hence evaluate $\log_2 3 \times \log_3 4 \times \log_4 5 \times \log_5 6 \times \log_6 7 \times \log_7 8$. **[5 Marks]**

Answer:

1. Change every logarithm to base 10: $\log_a b$ = log b / log a, $\log_b c$ = log c / log b, $\log_c d$ = log d / log c.
2. Product = (log b / log a) × (log c / log b) × (log d / log c) = log d / log a = $\log_a d$. Hence proved.
3. In the given product each denominator base cancels the next numerator number, so the whole chain collapses to $\log_2 8$.

4. $\log_2 8 = 3$, because $2^3 = 8$.

Answer: 3

Q10. If $a^x = b^y = c^z$ and $b^2 = ac$, prove that $y = 2xz / (x + z)$. **[5 Marks]**

Answer:

1. Let $a^x = b^y = c^z = k$. Then $a = k^{1/x}$, $b = k^{1/y}$ and $c = k^{1/z}$.
2. Given $b^2 = ac$, so $(k^{1/y})^2 = k^{1/x} \times k^{1/z}$, i.e. $k^{2/y} = k^{1/x + 1/z}$.
3. Equate the powers: $2/y = 1/x + 1/z = (x + z) / (xz)$.
4. Therefore $y = 2xz / (x + z)$. Hence proved.

Q11. If $\log x / (y - z) = \log y / (z - x) = \log z / (x - y)$, prove that (i) $xyz = 1$ and (ii) $x^x y^y z^z = 1$. **[6 Marks]**

Answer:

1. Let each ratio be equal to k. Then $\log x = k (y - z)$, $\log y = k (z - x)$ and $\log z = k (x - y)$.
2. (i) Add the three: $\log x + \log y + \log z = k [(y - z) + (z - x) + (x - y)] = k \times 0 = 0$.
3. So $\log (xyz) = 0 = \log 1$, which gives $xyz = 1$.
4. (ii) Multiply the three equations by x, y and z respectively and add: $x \log x + y \log y + z \log z = k [x(y - z) + y(z - x) + z(x - y)]$.
5. The bracket $= xy - xz + yz - xy + zx - yz = 0$. So $x \log x + y \log y + z \log z = 0$.
6. By the power law, $\log (x^x y^y z^z) = 0$, so $x^x y^y z^z = 1$. Hence proved.

Q12. If $a^2 + b^2 = 23ab$ $(a, b > 0)$, prove that $\log [(a + b)/5] = ½ (\log a + \log b)$. **[5 Marks]**

Answer:

1. Add 2ab to both sides: $a^2 + 2ab + b^2 = 25ab$, so $(a + b)^2 = 25ab$.
2. Divide by 25: $[(a + b)/5]^2 = ab$.
3. Take the positive square root: $(a + b)/5 = \sqrt{ab}$.
4. Take log: $\log [(a + b)/5] = ½ \log (ab) = ½ (\log a + \log b)$. Hence proved.

Q13. Given $\log 2 = 0.3010$ and $\log 3 = 0.4771$, find: (i) log 72 (ii) log 0.075 (iii) log of the cube root of 12 (iv) log 2.25. **[6 Marks]**

Answer:

- (i) $\log 72 = \log (2^3 \times 3^2) = 3(0.3010) + 2(0.4771) = 0.9030 + 0.9542 = 1.8572$.
- (ii) $0.075 = 3 / (4 \times 10)$. So $\log 0.075 = \log 3 - 2 \log 2 - \log 10 = 0.4771 - 0.6020 - 1 = -1.1249 = -2 + 0.8751$.
- (iii) $\log 12 = 2(0.3010) + 0.4771 = 1.0791$. So log of the cube root of 12 $= (1/3)(1.0791) = 0.3597$.
- (iv) $2.25 = 9/4$. So $\log 2.25 = 2 \log 3 - 2 \log 2 = 0.9542 - 0.6020 = 0.3522$.

Answer: (i) 1.8572 (ii) −1.1249 (characteristic −2, mantissa 0.8751) (iii) 0.3597 (iv) 0.3522

Q14. Given $\log 2 = 0.3010$ and $\log 3 = 0.4771$, find (i) the number of digits in 2^{100}, (ii) the number of digits in 6^{25}, (iii) the number of zeros after the decimal point in $(1/2)^{20}$, and (iv) the number of zeros after the decimal point in $(0.6)^{15}$. **[6 Marks]**

Answer:

- (i) $\log 2^{100} = 100 \times 0.3010 = 30.10$. Characteristic = 30, so the number of digits = 31.

- (ii) $\log 6 = 0.3010 + 0.4771 = 0.7781$. $\log 6^{25} = 25 \times 0.7781 = 19.4525$. Characteristic = 19, so the number of digits = 20.
- (iii) $\log (1/2)^{20} = -20 \times 0.3010 = -6.020 = -7 + 0.980$. Characteristic = −7, so the number of zeros = 7 − 1 = 6.
- (iv) $\log 0.6 = \log (6/10) = 0.7781 - 1 = -0.2219$. $\log (0.6)^{15} = -15 \times 0.2219 = -3.3285 = -4 + 0.6715$. Characteristic = −4, so the number of zeros = 4 − 1 = 3.

Answer: (i) 31 digits (ii) 20 digits (iii) 6 zeros (iv) 3 zeros

Q15. Solve the simultaneous equations: $\log x + \log y = 2$ and $x - y = 15$. **[5 Marks]**

Answer:

1. Here $x > 0$ and $y > 0$. From the first equation $\log (xy) = 2$, so $xy = 10^2 = 100$.
2. Use $(x + y)^2 = (x - y)^2 + 4xy = 225 + 400 = 625$. Since $x, y > 0$, $x + y = 25$.
3. Solve $x + y = 25$ and $x - y = 15$: $x = 20$ and $y = 5$.
4. Check: $\log 20 + \log 5 = \log 100 = 2$, and $20 - 5 = 15$. Both are satisfied.

Answer: $x = 20$, $y = 5$

Q16. Solve: $\log_4 (x + 3) - \log_4 (x - 1) = 2 - \log_4 8$. **[5 Marks]**

Answer:

1. Conditions: $x + 3 > 0$ and $x - 1 > 0$, so $x > 1$.
2. Write $2 = \log_4 4^2 = \log_4 16$.
3. Then $\log_4 [(x + 3)/(x - 1)] = \log_4 16 - \log_4 8 = \log_4 (16/8) = \log_4 2$.
4. Equate the arguments: $(x + 3)/(x - 1) = 2$, so $x + 3 = 2x - 2$ and $x = 5$.
5. $x = 5$ satisfies $x > 1$, so it is valid.

Answer: $x = 5$

Q17. Prove that $1/(1 + \log_a bc) + 1/(1 + \log_b ca) + 1/(1 + \log_c ab) = 1$. **[5 Marks]**

Answer:

1. Write $1 = \log_a a$. Then $1 + \log_a bc = \log_a a + \log_a bc = \log_a abc$.
2. So $1/(1 + \log_a bc) = 1 / \log_a abc = \log_{abc} a$ (reciprocal rule).
3. In the same way $1/(1 + \log_b ca) = \log_{abc} b$ and $1/(1 + \log_c ab) = \log_{abc} c$.
4. Sum $= \log_{abc} a + \log_{abc} b + \log_{abc} c = \log_{abc} (a \times b \times c) = \log_{abc} abc = 1$. Hence proved.

Q18. Solve $3^{2x-1} = 5^x$, given $\log 3 = 0.4771$ and $\log 5 = 0.6990$. **[5 Marks]**

Answer:

1. Take log on both sides: $(2x - 1) \log 3 = x \log 5$.
2. Expand: $2x \log 3 - \log 3 = x \log 5$, so $x (2 \log 3 - \log 5) = \log 3$.
3. Here $2 \log 3 - \log 5 = 0.9542 - 0.6990 = 0.2552$.
4. $x = 0.4771 / 0.2552 \approx 1.87$.

Answer: $x \approx 1.87$

Q19. Solve: $(\log x)^2 - 3 \log x + 2 = 0$. **[4 Marks]**

Answer:

1. Condition: $x > 0$. Put $t = \log x$. The equation becomes $t^2 - 3t + 2 = 0$.

2. Factorise: $(t - 1)(t - 2) = 0$, so $t = 1$ or $t = 2$.
3. If $\log x = 1$ then $x = 10$. If $\log x = 2$ then $x = 100$.
4. Both values are positive, so both are valid.

Answer: $x = 10$ or $x = 100$

Q20. Solve: $\log_2 x + \log_x 2 = 5/2$. **[5 Marks]**

Answer:

1. Conditions: $x > 0$ and $x \neq 1$.
2. Use $\log_x 2 = 1 / \log_2 x$ and put $t = \log_2 x$ ($t \neq 0$). Then $t + 1/t = 5/2$.
3. Multiply by $2t$: $2t^2 + 2 = 5t$, so $2t^2 - 5t + 2 = 0$, i.e. $(2t - 1)(t - 2) = 0$.
4. $t = 1/2$ or $t = 2$. So $x = 2^{1/2} = \sqrt{2}$ or $x = 2^2 = 4$.
5. Both values satisfy $x > 0$ and $x \neq 1$.

Answer: $x = \sqrt{2}$ or $x = 4$

Part 3: Quick Revision Summary

Formulas

- $a^x = N \leftrightarrow \log_a N = x \quad (a > 0, a \neq 1, N > 0)$
- $\log_a 1 = 0$, $\log_a a = 1$
- $\log_a (MN) = \log_a M + \log_a N$
- $\log_a (M/N) = \log_a M - \log_a N$
- $\log_a (M^n) = n \log_a M$
- $\log_a M = \log_b M / \log_b a$, $\log_a b = 1 / \log_b a$
- $\log_a b \times \log_b c = \log_a c$
- $\ln N = 2.3026 \log_{10} N$, $\log_{10} N = 0.4343 \ln N$

Characteristic and mantissa

- $N \geq 1$: characteristic = digits before the point − 1; $0 < N < 1$: characteristic = −(zeros after the point + 1).
- Number of digits in N = characteristic of log N + 1. The mantissa is always between 0 and 1.

Values to remember

- $\log 2 = 0.3010$, $\log 3 = 0.4771$, $\log 5 = 0.6990$, $\log 7 = 0.8451$, $\ln 10 = 2.303$

Part 4: How to Write Long Answers in the Exam

- For theory questions, begin with a clear definition, then give the formula, the proof or derivation, and finish with one numerical example.
- For "prove that" questions, start from the more complicated side, keep one common base throughout and end with "Hence proved".
- For equations, write the domain condition first and always reject invalid roots at the end.
- Show every step on a separate line and write the final answer clearly, preferably in a box.
- Substitute your answer back into the original equation to check it whenever time permits.