MATHEMATICS 1

Chapter: Logarithm

Prove That / Show That Type Questions with Solutions

For Engineering and Polytechnic Students | Exam-Oriented

Part 1: Standard Methods for Proving Logarithm Identities

"Prove that" and "show that" questions are very common in exams of this chapter. Almost all of them can be solved by one of the methods below. Decide the method first, then write the steps neatly.

- **Combine into one log.** When the left side is a sum of logs, use the product and quotient laws to write it as a single log. If the argument becomes 1, the answer is log 1 = 0.
- **Expand and cancel.** Write every log using log a, log b, log c and collect the coefficients. Terms that cancel give the result.
- **Take log of both sides.** For questions with powers such as $x^{y+z} y^{z+x} z^{x+y} = 1$, show that the log of the left side is 0.
- **Put the ratio equal to k.** If several ratios are equal, write log x = k (…), log y = k (…), log z = k (…) and add, or multiply by suitable numbers and add.
- **Put the powers equal to k.** If $a^x = b^y = c^z$, put each equal to k and write $a = k^{1/x}$, $b = k^{1/y}$, $c = k^{1/z}$.
- **Change to a common base.** For questions with logs of different bases, change everything to base 10 and cancel.
- **Make a perfect square.** For conditions like $a^2 + b^2 = 7ab$, add or subtract 2ab to get $(a + b)^2$ or $(a - b)^2$.
- **Use prime factors.** For numbers like 16/15 or 25/24, write each number using the primes 2, 3 and 5 and collect the coefficients.

Note: In this document "log" means $\log_{10}$ unless another base is written. Take all variables as positive so that every logarithm is defined.

Part 2: Direct Identities (Product, Quotient and Power Laws)

Q1. Prove that $\log (a^2/bc) + \log (b^2/ca) + \log (c^2/ab) = 0$.

Key idea: combine all three logs into one and show that the argument is 1.

Solution:

1. By the product law, LHS = $\log [(a^2/bc) \times (b^2/ca) \times (c^2/ab)]$.
2. Numerator = $a^2 \times b^2 \times c^2$. Denominator = $bc \times ca \times ab = a^2b^2c^2$.
3. So the argument = 1 and LHS = log 1 = 0 = RHS. Hence proved.

Another method: Expand each term using the quotient and power laws: (2 log a − log b − log c) + (2 log b − log c − log a) + (2 log c − log a − log b). The coefficient of each of log a, log b and log c is 2 − 1 − 1 = 0, so the sum is 0.

Q2. Prove that log (a/b) + log (b/c) + log (c/d) + log (d/a) = 0.

Key idea: after combining, every numerator cancels with a denominator.

Solution:

1. LHS = log [(a/b) × (b/c) × (c/d) × (d/a)].

2. The product = (a × b × c × d) / (b × c × d × a) = 1.
3. LHS = log 1 = 0. Hence proved.

Q3. Prove that log (ab/c) + log (bc/a) + log (ca/b) = log (abc).

Key idea: combine the three logs and simplify the product.

Solution:

1. LHS = log [(ab/c) × (bc/a) × (ca/b)].
2. Numerator = ab × bc × ca = $a^2b^2c^2$. Denominator = c × a × b = abc.
3. Argument = $a^2b^2c^2$ / abc = abc. So LHS = log (abc) = RHS. Hence proved.

Another method: Expand: (log a + log b − log c) + (log b + log c − log a) + (log c + log a − log b) = log a + log b + log c = log (abc).

Q4. Prove that log tan 1° + log tan 2° + log tan 3° + … + log tan 89° = 0.

Key idea: combine into one log; tan θ × tan (90° − θ) = 1, so the terms pair up to give 1.

Solution:

1. LHS = log (tan 1° × tan 2° × tan 3° × … × tan 89°).
2. Since tan (90° − θ) = cot θ, we have tan θ × tan (90° − θ) = tan θ × cot θ = 1.
3. Pair the terms: (1°, 89°), (2°, 88°), …, (44°, 46°). There are 44 pairs and each pair has product 1.
4. The middle term is tan 45° = 1.
5. So the whole product = 1 and LHS = log 1 = 0. Hence proved.

Q5. Prove that (i) log (1 + 1/1) + log (1 + 1/2) + log (1 + 1/3) + … + log (1 + 1/n) = log (n + 1), and (ii) log (1 − 1/2) + log (1 − 1/3) + … + log (1 − 1/n) = − log n.

Key idea: write each bracket as a fraction; the product telescopes.

Solution:

1. (i) 1 + 1/k = (k + 1)/k. So the sum = log [(2/1) × (3/2) × (4/3) × … × ((n + 1)/n)].
2. Every numerator cancels the next denominator, leaving n + 1. So the sum = log (n + 1). Hence proved.
3. (ii) 1 − 1/k = (k − 1)/k. So the sum = log [(1/2) × (2/3) × (3/4) × … × ((n − 1)/n)].
4. The product collapses to 1/n. So the sum = log (1/n) = − log n. Hence proved.

Q6. Prove that log 2 + log 4 + log 8 + … (n terms) = [n (n + 1) / 2] log 2.

Key idea: write each number as a power of 2 and use the power law.

Solution:

1. The terms are 2^1, 2^2, 2^3, …, 2^n.
2. By the power law, log 2^k = k log 2.
3. Sum = (1 + 2 + 3 + … + n) log 2.
4. Since 1 + 2 + 3 + … + n = n (n + 1) / 2, the sum = [n (n + 1) / 2] log 2. Hence proved.

Q7. Prove that log 2 + 16 log (16/15) + 12 log (25/24) + 7 log (81/80) = 1.

Key idea: write every number using the primes 2, 3, 5 and collect coefficients.

Solution:

1. Let p = log 2, q = log 3, r = log 5. Then 16 = 2^4, 15 = 3 × 5, 25 = 5^2, 24 = 2^3 × 3, 81 = 3^4, 80 = 2^4 × 5.
2. 16 log (16/15) = 16 (4p − q − r) = 64p − 16q − 16r.

3. $12 \log (25/24) = 12 (2r - 3p - q) = 24r - 36p - 12q$.
4. $7 \log (81/80) = 7 (4q - 4p - r) = 28q - 28p - 7r$.
5. Add all four terms (including p): coefficient of p = $1 + 64 - 36 - 28 = 1$; coefficient of q = $-16 - 12 + 28 = 0$; coefficient of r = $-16 + 24 - 7 = 1$.
6. LHS = $p + r = \log 2 + \log 5 = \log 10 = 1$ = RHS. Hence proved.

Part 3: Conditional Identities

Q8. If $a^2 + b^2 = 7ab$, prove that $\log [(a + b)/3] = \frac{1}{2} (\log a + \log b)$.

Key idea: add 2ab to make $(a + b)^2$.

Solution:

1. $(a + b)^2 = a^2 + b^2 + 2ab = 7ab + 2ab = 9ab$.
2. Divide by 9: $[(a + b)/3]^2 = ab$, so $(a + b)/3 = \sqrt{ab}$.
3. Take log: $\log [(a + b)/3] = \log \sqrt{ab} = \frac{1}{2} \log (ab) = \frac{1}{2} (\log a + \log b)$. Hence proved.

Q9. If $a^2 + b^2 = 11ab$ and $a > b > 0$, prove that $\log [(a - b)/3] = \frac{1}{2} (\log a + \log b)$.

Key idea: subtract 2ab to make $(a - b)^2$.

Solution:

1. $(a - b)^2 = a^2 + b^2 - 2ab = 11ab - 2ab = 9ab$.
2. So $[(a - b)/3]^2 = ab$. Since $a > b$, $(a - b)/3$ is positive and $(a - b)/3 = \sqrt{ab}$.
3. Take log: $\log [(a - b)/3] = \frac{1}{2} \log (ab) = \frac{1}{2} (\log a + \log b)$. Hence proved.

Q10. If $x^2 + 4y^2 = 12xy$, prove that $\log (x + 2y) = \log 4 + \frac{1}{2} (\log x + \log y)$.

Key idea: add 4xy to make $(x + 2y)^2$.

Solution:

1. $(x + 2y)^2 = x^2 + 4xy + 4y^2 = (x^2 + 4y^2) + 4xy = 12xy + 4xy = 16xy$.
2. So $x + 2y = 4\sqrt{xy}$ (positive root).
3. Take log: $\log (x + 2y) = \log 4 + \log \sqrt{xy} = \log 4 + \frac{1}{2} (\log x + \log y)$. Hence proved.

Q11. If $\log [(x + y)/2] = \frac{1}{2} (\log x + \log y)$, prove that $x = y$.

Key idea: remove the logs and use a perfect square.

Solution:

1. RHS = $\frac{1}{2} \log (xy) = \log \sqrt{xy}$.
2. Equate the arguments: $(x + y)/2 = \sqrt{xy}$, so $x + y = 2\sqrt{xy}$.
3. Rearrange: $x - 2\sqrt{xy} + y = 0$, i.e. $(\sqrt{x} - \sqrt{y})^2 = 0$.
4. So $\sqrt{x} = \sqrt{y}$ and $x = y$. Hence proved.

Q12. Prove that (i) if x, y, z are in G.P., then log x, log y, log z are in A.P.; (ii) if a, b, c are in G.P., then $\log_a N$, $\log_b N$, $\log_c N$ are in H.P.

Key idea: a G.P. has $b^2 = ac$; taking logs turns a product into a sum.

Solution:

1. (i) x, y, z in G.P. means $y^2 = xz$. Take log: $2 \log y = \log x + \log z$.
2. That is $\log y - \log x = \log z - \log y$, so log x, log y, log z are in A.P.
3. (ii) a, b, c in G.P., so by part (i), log a, log b, log c are in A.P. (with any base, in particular base N: $\log_N a$, $\log_N b$, $\log_N c$).
4. By the reciprocal rule, $\log_N a = 1 / \log_a N$, $\log_N b = 1 / \log_b N$ and $\log_N c = 1 / \log_c N$.

5. So $1 / \log_a N$, $1 / \log_b N$, $1 / \log_c N$ are in A.P. Numbers whose reciprocals are in A.P. are in H.P., hence $\log_a N$, $\log_b N$, $\log_c N$ are in H.P. Hence proved.

Part 4: Equal-Ratio Type Questions

Q13. If $\log x / (y - z) = \log y / (z - x) = \log z / (x - y)$, prove that (i) $xyz = 1$, (ii) $x^x y^y z^z = 1$, (iii) $x^{y+z} y^{z+x} z^{x+y} = 1$.

Key idea: put the common ratio equal to k. Add the equations for (i); multiply by suitable numbers and then add for (ii) and (iii).

Solution:

1. Let each ratio be k. Then $\log x = k (y - z)$ …(1), $\log y = k (z - x)$ …(2), $\log z = k (x - y)$ …(3).
2. (i) Add (1), (2), (3): $\log x + \log y + \log z = k [(y - z) + (z - x) + (x - y)] = k \times 0 = 0$.
3. So $\log (xyz) = 0 = \log 1$, which gives $xyz = 1$. Hence (i) is proved.
4. (ii) Multiply (1) by x, (2) by y, (3) by z and add: $x \log x + y \log y + z \log z = k [x(y - z) + y(z - x) + z(x - y)]$.
5. The bracket $= xy - xz + yz - xy + zx - yz = 0$. So $\log (x^x y^y z^z) = 0$, which gives $x^x y^y z^z = 1$. Hence (ii) is proved.
6. (iii) Multiply (1) by $(y + z)$, (2) by $(z + x)$, (3) by $(x + y)$ and add: $(y + z) \log x + (z + x) \log y + (x + y) \log z = k [(y^2 - z^2) + (z^2 - x^2) + (x^2 - y^2)] = 0$.
7. So $\log (x^{y+z} y^{z+x} z^{x+y}) = 0$, which gives $x^{y+z} y^{z+x} z^{x+y} = 1$. Hence (iii) is proved.

Q14. If $\log x / (b - c) = \log y / (c - a) = \log z / (a - b)$, prove that (i) $xyz = 1$, (ii) $x^a y^b z^c = 1$, (iii) $x^{b+c} y^{c+a} z^{a+b} = 1$.

Key idea: same method: common ratio k, then add, or multiply by a, b, c (or by $b + c$, $c + a$, $a + b$) and add.

Solution:

1. Let each ratio be k. Then $\log x = k (b - c)$ …(1), $\log y = k (c - a)$ …(2), $\log z = k (a - b)$ …(3).
2. (i) Add: $\log (xyz) = k [(b - c) + (c - a) + (a - b)] = 0$. So $xyz = 1$.
3. (ii) Multiply (1), (2), (3) by a, b, c respectively and add: $a \log x + b \log y + c \log z = k [a(b - c) + b(c - a) + c(a - b)]$.
4. The bracket $= ab - ac + bc - ab + ca - bc = 0$. So $\log (x^a y^b z^c) = 0$ and $x^a y^b z^c = 1$.
5. (iii) Multiply (1), (2), (3) by $(b + c)$, $(c + a)$, $(a + b)$ respectively and add: the right side $= k [(b^2 - c^2) + (c^2 - a^2) + (a^2 - b^2)] = 0$.
6. So $\log (x^{b+c} y^{c+a} z^{a+b}) = 0$ and $x^{b+c} y^{c+a} z^{a+b} = 1$. Hence proved.

Part 5: Identities with Powers of Logarithms

Q15. Prove that $x^{\log y - \log z} \times y^{\log z - \log x} \times z^{\log x - \log y} = 1$.

Key idea: take log of the whole expression and show that it is 0.

Solution:

1. Let E be the left side. Then $\log E = (\log y - \log z) \log x + (\log z - \log x) \log y + (\log x - \log y) \log z$.
2. Expand: $\log x \log y - \log x \log z + \log y \log z - \log y \log x + \log z \log x - \log z \log y$.
3. The terms cancel in pairs ($\log x \log y$ with $- \log y \log x$, and so on). So $\log E = 0$.
4. Therefore $E = 1$. Hence proved.

Q16. Prove that (i) $x^{\log y} = y^{\log x}$, and (ii) a raised to the power ($\log_b c$) equals c raised to the power ($\log_b a$).

Key idea: take the log of both sides; the two answers turn out to be the same product.

Solution:

1. (i) Take log of the left side: $\log (x^{\log y}) = (\log y)(\log x)$. Take log of the right side: $\log (y^{\log x}) = (\log x)(\log y)$.
2. Both are equal, so $x^{\log y} = y^{\log x}$. Hence (i) is proved.
3. (ii) Take log to the base b of the first expression: $(\log_b c) \times (\log_b a)$. Take log to the base b of the second expression: $(\log_b a) \times (\log_b c)$.
4. Both are equal, so the two expressions are equal. Hence (ii) is proved.

Part 6: Change of Base and Chain Type Questions

Q17. Prove that $\log_a N / \log_{ab} N = 1 + \log_a b$.

Key idea: change both logs to base 10.

Solution:

1. $\log_a N = \log N / \log a$ and $\log_{ab} N = \log N / \log (ab)$.
2. LHS = $(\log N / \log a) \times (\log (ab) / \log N) = \log (ab) / \log a$.
3. $\log (ab) = \log a + \log b$, so LHS = $(\log a + \log b) / \log a = 1 + \log b / \log a = 1 + \log_a b$ = RHS. Hence proved.

Q18. Prove that $\log_a x \times \log_b y = \log_b x \times \log_a y$.

Key idea: change all four logs to base 10.

Solution:

1. LHS = $(\log x / \log a) \times (\log y / \log b) = (\log x \times \log y) / (\log a \times \log b)$.
2. RHS = $(\log x / \log b) \times (\log y / \log a) = (\log x \times \log y) / (\log a \times \log b)$.
3. LHS = RHS. Hence proved.

Q19. Prove that the logarithm of x^n to the base a^m is equal to $(n/m) \log_a x$. Hence find the logarithm of 16 to the base 8.

Key idea: change to base 10 and use the power law on both numerator and denominator.

Solution:

1. The logarithm of x^n to the base $a^m = \log x^n / \log a^m$.
2. By the power law this equals $n \log x / (m \log a) = (n/m) \times (\log x / \log a) = (n/m) \log_a x$. Hence proved.
3. For the example, $16 = 2^4$ and $8 = 2^3$. So the logarithm of 16 to the base 8 = $(4/3) \log_2 2 = 4/3$.

Q20. Prove that $1/\log_3 60 + 1/\log_4 60 + 1/\log_5 60 = 1$.

Key idea: use the reciprocal rule $1 / \log_a N = \log_N a$, then combine.

Solution:

1. LHS = $\log_{60} 3 + \log_{60} 4 + \log_{60} 5$.
2. By the product law, LHS = $\log_{60} (3 \times 4 \times 5) = \log_{60} 60$.
3. $\log_{60} 60 = 1$. Hence proved.

Q21. If $a = \log_{24} 12$, $b = \log_{36} 24$ and $c = \log_{48} 36$, prove that $1 + abc = 2bc$.

Key idea: change to base 10 so that the terms in abc cancel.

Solution:

1. $a = \log 12 / \log 24$, $b = \log 24 / \log 36$, $c = \log 36 / \log 48$.
2. $abc = (\log 12 / \log 24) \times (\log 24 / \log 36) \times (\log 36 / \log 48) = \log 12 / \log 48$.
3. $bc = (\log 24 / \log 36) \times (\log 36 / \log 48) = \log 24 / \log 48$.
4. LHS = $1 + abc = (\log 48 + \log 12) / \log 48 = \log 576 / \log 48$, because $48 \times 12 = 576$.
5. RHS = $2bc = 2 \log 24 / \log 48 = \log 24^2 / \log 48 = \log 576 / \log 48$.
6. LHS = RHS. Hence proved.

Q22. If $x = 1 + \log_a bc$, $y = 1 + \log_b ca$ and $z = 1 + \log_c ab$, prove that $xyz = xy + yz + zx$.

Key idea: show that $1/x + 1/y + 1/z = 1$ and then multiply by xyz.

Solution:

1. $x = \log_a a + \log_a bc = \log_a abc = \log (abc) / \log a$. Let $L = \log (abc)$.
2. So $x = L / \log a$. In the same way $y = L / \log b$ and $z = L / \log c$.
3. Therefore $1/x = \log a / L$, $1/y = \log b / L$ and $1/z = \log c / L$.
4. $1/x + 1/y + 1/z = (\log a + \log b + \log c) / L = \log (abc) / L = 1$.
5. Multiply both sides by xyz: $yz + zx + xy = xyz$. Hence proved.

Part 7: Equal-Power Type Questions

Q23. If $a^x = b^y = (ab)^z$, prove that $z = xy / (x + y)$.

Key idea: put each expression equal to k and write a, b and ab as powers of k.

Solution:

1. Let $a^x = b^y = (ab)^z = k$. Then $a = k^{1/x}$, $b = k^{1/y}$ and $ab = k^{1/z}$.
2. So $k^{1/x} \times k^{1/y} = k^{1/z}$, i.e. $k^{1/x + 1/y} = k^{1/z}$.
3. Equate the powers: $1/x + 1/y = 1/z$, so $(x + y) / xy = 1/z$.
4. Hence $z = xy / (x + y)$. Hence proved.

Q24. If $2^a = 3^b = 12^c$, prove that $1/c = 2/a + 1/b$.

Key idea: write $12 = 2^2 \times 3$ and express everything as powers of k.

Solution:

1. Let $2^a = 3^b = 12^c = k$. Then $2 = k^{1/a}$, $3 = k^{1/b}$ and $12 = k^{1/c}$.
2. Since $12 = 2^2 \times 3$, we get $k^{1/c} = (k^{1/a})^2 \times k^{1/b} = k^{2/a + 1/b}$.
3. Equate the powers: $1/c = 2/a + 1/b$. Hence proved.

Q25. If $a^x = b^y = c^z = d^w$ and $abcd = 1$, prove that $1/x + 1/y + 1/z + 1/w = 0$.

Key idea: write a, b, c, d as powers of k and use $abcd = 1$.

Solution:

1. Let $a^x = b^y = c^z = d^w = k$ $(k \neq 1)$. Then $a = k^{1/x}$, $b = k^{1/y}$, $c = k^{1/z}$ and $d = k^{1/w}$.
2. $abcd = k^{1/x + 1/y + 1/z + 1/w}$.
3. Given $abcd = 1 = k^0$, so $k^{1/x + 1/y + 1/z + 1/w} = k^0$.
4. Equate the powers: $1/x + 1/y + 1/z + 1/w = 0$. Hence proved.

Part 8: Practice Questions with Hints

Try to prove each of these on your own. A short hint is given below every question.

Q26. Prove that log (bc/a^2) + log (ca/b^2) + log (ab/c^2) = 0.
Hint: Combine into one log; the argument becomes 1.

Q27. If $a^2 + b^2 = 14ab$, prove that log [(a + b)/4] = ½ (log a + log b).
Hint: $(a + b)^2 = 16ab$.

Q28. If $a^2 + b^2 = 6ab$, prove that log [(a + b)/(2√2)] = ½ (log a + log b).
Hint: $(a + b)^2 = 8ab$, so $[(a + b)/(2\sqrt{2})]^2 = ab$.

Q29. If $a^x = b^y = c^z$ and abc = 1, prove that 1/x + 1/y + 1/z = 0.
Hint: Put each equal to k; abc = $k^{1/x + 1/y + 1/z} = k^0$.

Q30. If $3^a = 5^b = 15^c$, prove that 1/c = 1/a + 1/b.
Hint: Use 15 = 3 × 5 and write everything as powers of k.

Q31. Prove that $\log_a x \times \log_x y \times \log_y a = 1$.
Hint: Change every log to base 10; everything cancels.

Q32. Prove that $1/\log_2 100 + 1/\log_5 100 = 1/2$.
Hint: Use the reciprocal rule to get $\log_{100} 2 + \log_{100} 5 = \log_{100} 10$.

Q33. Prove that log 3 + log 9 + log 27 + … (n terms) = [n (n + 1) / 2] log 3.
Hint: Write the terms as $3^1, 3^2, 3^3$, … and use the power law.

Q34. If $\log_a (ab) = x$, prove that $\log_b (ab) = x / (x - 1)$.
Hint: $\log_a (ab) = 1 + \log_a b$, so $\log_a b = x - 1$; then $\log_b a = 1/(x - 1)$ and $\log_b (ab) = 1 + \log_b a$.