

MATHEMATICS 1

Chapter: Quadratic Equation

Broad (Long Answer) Questions with Complete Solutions

Part 1: Nature of Roots, Conditions and Inequalities

Q1. Find the values of k for which the equation $x^2 + 2(k + 2)x + 9k = 0$ has equal roots. Find the equal roots in each case.

Solution:

1. Here $a = 1$, $b = 2(k + 2)$, $c = 9k$.
2. Equal roots need $D = 0$: $[2(k + 2)]^2 - 4(1)(9k) = 0$.
3. Divide by 4: $(k + 2)^2 - 9k = 0$, so $k^2 + 4k + 4 - 9k = 0$.
4. $k^2 - 5k + 4 = 0$, so $(k - 1)(k - 4) = 0$, so $k = 1$ or $k = 4$.
5. For $k = 1$: $x^2 + 6x + 9 = 0$, i.e. $(x + 3)^2 = 0$, so $x = -3$ (repeated).
6. For $k = 4$: $x^2 + 12x + 36 = 0$, i.e. $(x + 6)^2 = 0$, so $x = -6$ (repeated).

Ans. $k = 1$ (roots $-3, -3$) or $k = 4$ (roots $-6, -6$).

Q2. If the roots of $ax^2 + 2bx + c = 0$ and $bx^2 - 2\sqrt{ac}x + b = 0$ are simultaneously real, prove that $b^2 = ac$.

Solution:

1. For the first equation, the roots are real, so $D_1 \geq 0$: $(2b)^2 - 4ac \geq 0$, so $b^2 \geq ac$... (i)
2. For the second equation, the roots are real, so $D_2 \geq 0$: $[-2\sqrt{ac}]^2 - 4(b)(b) \geq 0$.
3. That is $4ac - 4b^2 \geq 0$, so $ac \geq b^2$... (ii)
4. From (i) $b^2 \geq ac$ and from (ii) $b^2 \leq ac$. Both can hold together only if $b^2 = ac$.

Ans. $b^2 = ac$. Hence proved.

Q3. Prove that the roots of $(a^2 + b^2)x^2 + 2(ac + bd)x + (c^2 + d^2) = 0$ are imaginary unless $ad = bc$, in which case they are equal.

Solution:

1. $A = a^2 + b^2$, $B = 2(ac + bd)$, $C = c^2 + d^2$.
2. $D = 4(ac + bd)^2 - 4(a^2 + b^2)(c^2 + d^2)$.
3. Expand: $(ac + bd)^2 = a^2c^2 + 2abcd + b^2d^2$, and $(a^2 + b^2)(c^2 + d^2) = a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2$.
4. $D = 4[2abcd - a^2d^2 - b^2c^2] = -4(ad - bc)^2$.
5. A square is never negative, so $D \leq 0$. If $ad \neq bc$ then $D < 0$ and the roots are imaginary.
6. If $ad = bc$, then $D = 0$ and the roots are real and equal.

Ans. $D = -4(ad - bc)^2$. Hence proved.

Q4. Solve the inequalities: (i) $x^2 - 5x + 6 > 0$ (ii) $x^2 - 3x - 10 \geq 0$ (iii) $2x^2 + x - 6 < 0$.

Solution:

1. **Method:** factorise, find the roots, and use the fact that for a > 0 the expression is negative between the roots and positive outside them.
2. **(i)** $(x - 2)(x - 3) > 0$. The roots are 2 and 3. Positive outside the roots, so $x < 2$ or $x > 3$.
3. **(ii)** $(x - 5)(x + 2) \geq 0$. The roots are 5 and -2 . Non-negative outside (and at) the roots, so $x \leq -2$ or $x \geq 5$.
4. **(iii)** $(2x - 3)(x + 2) < 0$. The roots are $3/2$ and -2 . Negative between the roots, so $-2 < x < 3/2$.
5. Check (iii) with $x = 0$: $2(0) + 0 - 6 = -6 < 0$ ✓ (0 lies between -2 and $3/2$).

Ans. (i) $x < 2$ or $x > 3$ (ii) $x \leq -2$ or $x \geq 5$ (iii) $-2 < x < 3/2$.

Q5. Using the method of completing the square, find (a) the minimum value of $x^2 - 6x + 13$, (b) the maximum value of $5 + 4x - x^2$, and (c) the maximum product of two numbers whose sum is 20.

Solution:

1. **(a)** $x^2 - 6x + 13 = (x^2 - 6x + 9) + 4 = (x - 3)^2 + 4$.
2. $(x - 3)^2 \geq 0$, so the least value is 4, when $x = 3$.
3. **(b)** $5 + 4x - x^2 = 5 - (x^2 - 4x) = 5 - [(x - 2)^2 - 4] = 9 - (x - 2)^2$.
4. $(x - 2)^2 \geq 0$, so the greatest value is 9, when $x = 2$.
5. **(c)** Let the numbers be x and $20 - x$. Product $= x(20 - x) = 20x - x^2 = 100 - (x - 10)^2$.
6. The greatest value is 100, when $x = 10$, so both numbers are 10.

Ans. (a) Minimum = 4 at $x = 3$ (b) Maximum = 9 at $x = 2$ (c) Maximum product = 100 (numbers 10 and 10).

Part 2: Relations Between the Roots

Q6. If α and β are the roots of $x^2 - 6x + 1 = 0$, find (i) $\alpha^2 + \beta^2$, (ii) $\alpha^3 + \beta^3$, (iii) $\alpha^4 + \beta^4$, (iv) $|\alpha - \beta|$.

Solution:

1. $\alpha + \beta = 6$ and $\alpha\beta = 1$.
2. **(i)** $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = 36 - 2 = 34$.
3. **(ii)** $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta) = 216 - 3(1)(6) = 198$.
4. **(iii)** $\alpha^4 + \beta^4 = (\alpha^2 + \beta^2)^2 - 2(\alpha\beta)^2 = 34^2 - 2 = 1156 - 2 = 1154$.
5. **(iv)** $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta = 36 - 4 = 32$, so $|\alpha - \beta| = \sqrt{32} = 4\sqrt{2}$.
6. Check: the roots are $3 \pm 2\sqrt{2}$, so their difference is $4\sqrt{2}$ ✓

Ans. (i) 34 (ii) 198 (iii) 1154 (iv) $4\sqrt{2}$.

Q7. If α and β are the roots of $x^2 - px + q = 0$, form the equation whose roots are $\alpha + 1/\beta$ and $\beta + 1/\alpha$.

Solution:

1. $\alpha + \beta = p$ and $\alpha\beta = q$.
2. Sum of new roots $= \alpha + 1/\beta + \beta + 1/\alpha = (\alpha + \beta) + (\alpha + \beta)/\alpha\beta = p + p/q = p(q + 1)/q$.
3. Product of new roots $= (\alpha + 1/\beta)(\beta + 1/\alpha) = \alpha\beta + 1 + 1 + 1/(\alpha\beta) = q + 2 + 1/q = (q + 1)^2/q$.
4. Equation: $x^2 - [p(q + 1)/q]x + (q + 1)^2/q = 0$.
5. Multiply by q : $qx^2 - p(q + 1)x + (q + 1)^2 = 0$.
6. Check with $x^2 - 3x + 2 = 0$ (roots 1, 2; $p = 3$, $q = 2$): new roots are $1 + 1/2 = 3/2$ and $2 + 1 = 3$. The formula gives $2x^2 - 9x + 9 = 0$, i.e. $(2x - 3)(x - 3) = 0$ ✓

Ans. $qx^2 - p(q + 1)x + (q + 1)^2 = 0$.

Q8. If α and β are the roots of $ax^2 + bx + c = 0$, prove that $1/(a\alpha + b)^2 + 1/(a\beta + b)^2 = (b^2 - 2ac)/(a^2c^2)$.

Solution:

1. Since $\alpha + \beta = -b/a$, we have $b = -a(\alpha + \beta)$.
2. So $a\alpha + b = a\alpha - a\alpha - a\beta = -a\beta$. Similarly $a\beta + b = -a\alpha$.
3. LHS $= 1/(a^2\beta^2) + 1/(a^2\alpha^2) = (\alpha^2 + \beta^2)/(a^2\alpha^2\beta^2)$.
4. $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = b^2/a^2 - 2c/a = (b^2 - 2ac)/a^2$.
5. $\alpha^2\beta^2 = c^2/a^2$, so $a^2\alpha^2\beta^2 = c^2$.
6. LHS $= [(b^2 - 2ac)/a^2] \div c^2 = (b^2 - 2ac)/(a^2c^2)$.
7. Check with $x^2 - 3x + 2 = 0$: $a = 1$, $b = -3$, $c = 2$. $a\alpha + b = -2$ and -1 , so $1/4 + 1 = 5/4$, and $(9 - 4)/4 = 5/4$ ✓

Ans. Hence proved.

Q9. If the ratio of the roots of $ax^2 + bx + c = 0$ is $p : q$, prove that $pqb^2 = (p + q)^2 ac$.

Solution:

1. Let the roots be pk and qk , where $k \neq 0$.
2. Sum: $(p + q)k = -b/a$, so $k = -b/[a(p + q)]$... (i)
3. Product: $pq k^2 = c/a$... (ii)
4. Substitute (i) in (ii): $pq \cdot b^2/[a^2(p + q)^2] = c/a$.
5. Multiply by $a^2(p + q)^2$: $pqb^2 = (p + q)^2 ac$.
6. Check with $x^2 - 3x + 2 = 0$ (roots 1 and 2, ratio 1 : 2): $pqb^2 = 2 \times 9 = 18$ and $(p + q)^2 ac = 9 \times 2 = 18$ ✓

Ans. $pqb^2 = (p + q)^2 ac$. Hence proved.

Q10. Find the values of k for which the equations $x^2 - 2x - 3 = 0$ and $x^2 + kx - 6 = 0$ have a common root.

Solution:

1. Solve the first equation: $(x - 3)(x + 1) = 0$, so its roots are 3 and -1 .
2. A common root must satisfy the second equation as well.
3. Try $x = 3$: $9 + 3k - 6 = 0$, so $3k = -3$, so $k = -1$. The second equation is $x^2 - x - 6 = (x - 3)(x + 2) = 0$, and 3 is indeed a root ✓
4. Try $x = -1$: $1 - k - 6 = 0$, so $k = -5$. The second equation is $x^2 - 5x - 6 = (x - 6)(x + 1) = 0$, and -1 is indeed a root ✓

Ans. $k = -1$ (common root 3) or $k = -5$ (common root -1).

Part 3: Equations Reducible to Quadratic Form

Q11. Solve $2(x^2 + 1/x^2) - 9(x + 1/x) + 14 = 0$.

Solution:

1. Put $y = x + 1/x$. Then $y^2 = x^2 + 2 + 1/x^2$, so $x^2 + 1/x^2 = y^2 - 2$.
2. The equation becomes $2(y^2 - 2) - 9y + 14 = 0$, so $2y^2 - 9y + 10 = 0$.
3. Factorise: $(2y - 5)(y - 2) = 0$, so $y = 5/2$ or $y = 2$.
4. Case $y = 2$: $x + 1/x = 2$, so $x^2 - 2x + 1 = 0$, so $(x - 1)^2 = 0$, so $x = 1$ (repeated).
5. Case $y = 5/2$: $x + 1/x = 5/2$, so $2x^2 - 5x + 2 = 0$, so $(2x - 1)(x - 2) = 0$, so $x = 1/2$ or $x = 2$.
6. Check $x = 2$: $2(4 + 1/4) - 9(5/2) + 14 = 8.5 - 22.5 + 14 = 0$ ✓

Ans. $x = 1$ (repeated), $x = 2$, $x = 1/2$.

Q12. Solve (a) $4^x - 6 \cdot 2^x + 8 = 0$ (b) $9^x - 4 \cdot 3^x + 3 = 0$.

Solution:

1. **(a)** Write $4^x = (2^x)^2$. Put $y = 2^x$: $y^2 - 6y + 8 = 0$, so $(y - 2)(y - 4) = 0$, so $y = 2$ or $y = 4$.
2. $2^x = 2$ gives $x = 1$; $2^x = 4$ gives $x = 2$.
3. Check $x = 2$: $16 - 24 + 8 = 0$ ✓
4. **(b)** Write $9^x = (3^x)^2$. Put $y = 3^x$: $y^2 - 4y + 3 = 0$, so $(y - 1)(y - 3) = 0$, so $y = 1$ or $y = 3$.
5. $3^x = 1$ gives $x = 0$; $3^x = 3$ gives $x = 1$.
6. Check $x = 0$: $1 - 4 + 3 = 0$ ✓

Ans. (a) $x = 1$ or $x = 2$ (b) $x = 0$ or $x = 1$.

Q13. Solve (a) $x^{2/3} + x^{1/3} - 6 = 0$ (b) $2x - 7\sqrt{x} + 3 = 0$.

Solution:

1. **(a)** Put $y = x^{1/3}$, so $x^{2/3} = y^2$. Then $y^2 + y - 6 = 0$, so $(y + 3)(y - 2) = 0$, so $y = -3$ or $y = 2$.
2. $x^{1/3} = 2$ gives $x = 8$; $x^{1/3} = -3$ gives $x = -27$.

3. Check $x = -27$: $(-27)^{2/3} = 9$ and $(-27)^{1/3} = -3$, so $9 - 3 - 6 = 0$ ✓
4. (b) Put $y = \sqrt{x}$, so $x = y^2$. Then $2y^2 - 7y + 3 = 0$, so $(2y - 1)(y - 3) = 0$, so $y = 1/2$ or $y = 3$.
5. $y = 1/2$ gives $x = 1/4$; $y = 3$ gives $x = 9$.
6. Check $x = 9$: $18 - 21 + 3 = 0$ ✓; $x = 1/4$: $1/2 - 7/2 + 3 = 0$ ✓

Ans. (a) $x = 8$ or $x = -27$ (b) $x = 1/4$ or $x = 9$.

Part 4: Applications

Q14. A boat can travel at 15 km/h in still water. It goes 30 km upstream and returns to the starting point in 4 hours 30 minutes. Find the speed of the stream.

Solution:

1. Let the speed of the stream = x km/h.
2. Upstream speed = $(15 - x)$ km/h and downstream speed = $(15 + x)$ km/h.
3. Total time = $30/(15 - x) + 30/(15 + x) = 4.5$ hours.
4. Combine: $30[(15 + x) + (15 - x)] / [(15 - x)(15 + x)] = 4.5$, so $900/(225 - x^2) = 4.5$.
5. $225 - x^2 = 200$, so $x^2 = 25$, so $x = 5$ (the negative value is rejected).
6. Check: upstream time = $30/10 = 3$ h and downstream time = $30/20 = 1.5$ h, total 4.5 h ✓

Ans. Speed of the stream = 5 km/h.

Q15. Two water taps together can fill a tank in 6 hours. The tap with the longer diameter takes 5 hours less than the tap with the smaller diameter to fill the tank separately. Find the time each tap takes to fill the tank.

Solution:

1. Let the smaller tap take x hours alone. Then the larger tap takes $(x - 5)$ hours alone.
2. In one hour they fill $1/x$ and $1/(x - 5)$ of the tank. Together: $1/x + 1/(x - 5) = 1/6$.
3. Combine: $(2x - 5)/[x(x - 5)] = 1/6$, so $6(2x - 5) = x(x - 5)$.
4. $12x - 30 = x^2 - 5x$, so $x^2 - 17x + 30 = 0$.
5. $(x - 15)(x - 2) = 0$, so $x = 15$ or $x = 2$.
6. $x = 2$ is rejected, since then $(x - 5)$ would be negative. So $x = 15$ and $x - 5 = 10$.
7. Check: $1/15 + 1/10 = 2/30 + 3/30 = 5/30 = 1/6$ ✓

Ans. The smaller tap takes 15 hours and the larger tap takes 10 hours.

Q16. A rectangular park is 40 m long and 30 m wide. A path of uniform width runs inside the park along its boundary. If the area of the path is 600 m², find the width of the path.

Solution:

1. Let the width of the path = x m.
2. The inner rectangle (without the path) has length $(40 - 2x)$ and width $(30 - 2x)$.
3. Area of park = $40 \times 30 = 1200$ m². Area of path = $1200 - (40 - 2x)(30 - 2x) = 600$.

4. So $(40 - 2x)(30 - 2x) = 600$.
5. Expand: $1200 - 140x + 4x^2 = 600$, so $4x^2 - 140x + 600 = 0$, so $x^2 - 35x + 150 = 0$.
6. $(x - 5)(x - 30) = 0$, so $x = 5$ or $x = 30$.
7. $x = 30$ is impossible because the width of the park is only 30 m. So $x = 5$.
8. Check: inner rectangle = $30 \times 20 = 600 \text{ m}^2$, so the path area = $1200 - 600 = 600 \text{ m}^2$ ✓

Ans. Width of the path = 5 m.

Q17. A ball is thrown upwards so that its height (in metres) after t seconds is $h = 20t - 5t^2$. Find (a) the times at which the height is 15 m, (b) the maximum height and the time at which it is reached, (c) the total time of flight.

Solution:

1. (a) Put $h = 15$: $20t - 5t^2 = 15$, so $t^2 - 4t + 3 = 0$, so $(t - 1)(t - 3) = 0$, so $t = 1 \text{ s}$ or $t = 3 \text{ s}$ (once going up, once coming down).
2. (b) $h = -5(t^2 - 4t) = -5[(t - 2)^2 - 4] = 20 - 5(t - 2)^2$.
3. The greatest value is 20 m, when $t = 2 \text{ s}$.
4. (c) The ball is on the ground when $h = 0$: $5t(4 - t) = 0$, so $t = 0$ or $t = 4$.

Ans. (a) $t = 1 \text{ s}$ and $t = 3 \text{ s}$ (b) Maximum height = 20 m at $t = 2 \text{ s}$ (c) Time of flight = 4 s.

Q18. Two resistors connected in series have a total resistance of 10Ω . When they are connected in parallel, the equivalent resistance is 2.4Ω . Find the two resistances.

Solution:

1. Let the resistances be R_1 and R_2 . Series: $R_1 + R_2 = 10 \dots$ (i)
2. Parallel: $R_1 R_2 / (R_1 + R_2) = 2.4$. Using (i): $R_1 R_2 / 10 = 2.4$, so $R_1 R_2 = 24 \dots$ (ii)
3. R_1 and R_2 are the roots of $t^2 - (\text{sum})t + (\text{product}) = 0$, i.e. $t^2 - 10t + 24 = 0$.
4. $(t - 4)(t - 6) = 0$, so $t = 4$ or $t = 6$.
5. Check: $4 + 6 = 10 \Omega$ and $(4 \times 6) / 10 = 2.4 \Omega$ ✓

Ans. The resistances are 4Ω and 6Ω .