

MATHEMATICS 1

Chapter: Quadratic Equation

Broad (Long Answer) Questions with Complete Solutions: Set 2

Part 1: Nature and Conditions of Roots

Q1. If the roots of $(a^2 + b^2)x^2 - 2b(a + c)x + (b^2 + c^2) = 0$ are real and equal, prove that a, b, c are in G.P.

Solution:

1. Equal roots need $D = 0$: $4b^2(a + c)^2 - 4(a^2 + b^2)(b^2 + c^2) = 0$.
2. Divide by 4: $b^2(a + c)^2 = (a^2 + b^2)(b^2 + c^2)$.
3. Left side: $a^2b^2 + 2ab^2c + b^2c^2$.
4. Right side: $a^2b^2 + a^2c^2 + b^4 + b^2c^2$.
5. Cancel a^2b^2 and b^2c^2 from both sides: $2ab^2c = a^2c^2 + b^4$.
6. Rearrange: $b^4 - 2ab^2c + a^2c^2 = 0$, i.e. $(b^2 - ac)^2 = 0$.
7. So $b^2 = ac$, which is exactly the condition for a, b, c to be in G.P.

Ans. $b^2 = ac$, so a, b, c are in G.P. Hence proved.

Q2. Find the values of p for which the equation $x^2 - 2px + 7p - 12 = 0$ has (i) equal roots, (ii) real roots, (iii) no real roots.

Solution:

1. Here $a = 1$, $b = -2p$, $c = 7p - 12$.
2. $D = 4p^2 - 4(7p - 12) = 4(p^2 - 7p + 12) = 4(p - 3)(p - 4)$.
3. **(i) Equal roots:** $D = 0$, so $p = 3$ or $p = 4$.
4. For $p = 3$: $x^2 - 6x + 9 = (x - 3)^2$, so $x = 3$. For $p = 4$: $x^2 - 8x + 16 = (x - 4)^2$, so $x = 4$.
5. **(ii) Real roots:** $D \geq 0$, so $(p - 3)(p - 4) \geq 0$, so $p \leq 3$ or $p \geq 4$.
6. **(iii) No real roots:** $D < 0$, so $(p - 3)(p - 4) < 0$, so $3 < p < 4$.

Ans. (i) $p = 3$ or 4 (ii) $p \leq 3$ or $p \geq 4$ (iii) $3 < p < 4$.

Q3. Prove that if $a + b + c = 0$, then 1 is a root of $ax^2 + bx + c = 0$ and the other root is c/a . Similarly, if $a - b + c = 0$, the roots are -1 and $-c/a$. Hence solve (i) $3x^2 - 5x + 2 = 0$, (ii) $7x^2 + 2x - 9 = 0$, (iii) $2x^2 + 5x + 3 = 0$.

Solution:

1. Put $x = 1$ in $ax^2 + bx + c$: the value is $a + b + c = 0$, so $x = 1$ is a root.
2. The product of the roots is c/a , so if one root is 1, the other root is c/a .
3. Put $x = -1$: the value is $a - b + c$. If $a - b + c = 0$, then $x = -1$ is a root.
4. The product of the roots is c/a , so $(-1) \times (\text{other root}) = c/a$, and the other root is $-c/a$.
5. **(i)** $3 - 5 + 2 = 0$, so $a + b + c = 0$. Roots: 1 and $2/3$.

6. **(ii)** $7 + 2 - 9 = 0$, so $a + b + c = 0$. Roots: 1 and $-9/7$.
7. **(iii)** $2 - 5 + 3 = 0$, so $a - b + c = 0$. Roots: -1 and $-3/2$.
8. Check (ii): sum of roots $= 1 - 9/7 = -2/7 = -b/a$ ✓

Ans. (i) 1, $2/3$ (ii) 1, $-9/7$ (iii) -1 , $-3/2$.

Q4. For the equation $x^2 - 6x + k = 0$, find the values of k for which (i) both roots are positive, (ii) the roots are of opposite signs, (iii) both roots are negative.

Solution:

1. Sum of roots $= 6$ and product of roots $= k$. Discriminant $D = 36 - 4k$.
2. **(i) Both positive:** the roots must be real ($D \geq 0$, so $k \leq 9$), the sum must be positive ($6 > 0$, always true), and the product must be positive ($k > 0$). So $0 < k \leq 9$.
3. **(ii) Opposite signs:** the product must be negative, so $k < 0$. Then $D = 36 - 4k > 0$ automatically, so the roots are real.
4. **(iii) Both negative:** the sum of the roots would have to be negative, but the sum is $6 > 0$. So this is impossible.
5. Check (i) with $k = 5$: roots 1 and 5, both positive ✓. Check (ii) with $k = -7$: roots 7 and -1 ✓

Ans. (i) $0 < k \leq 9$ (ii) $k < 0$ (iii) no value of k .

Q5. If the sum of the roots of $ax^2 + bx + c = 0$ is equal to the sum of the squares of their reciprocals, prove that bc^2, ca^2, ab^2 are in A.P.

Solution:

1. Let the roots be α and β . Given $\alpha + \beta = 1/\alpha^2 + 1/\beta^2$.
2. $RHS = (\alpha^2 + \beta^2)/(\alpha\beta)^2$. Use $\alpha + \beta = -b/a$ and $\alpha\beta = c/a$.
3. $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = b^2/a^2 - 2c/a = (b^2 - 2ac)/a^2$.
4. $(\alpha\beta)^2 = c^2/a^2$, so $RHS = [(b^2 - 2ac)/a^2] \div [c^2/a^2] = (b^2 - 2ac)/c^2$.
5. Equate: $-b/a = (b^2 - 2ac)/c^2$, so $-bc^2 = a(b^2 - 2ac) = ab^2 - 2a^2c$.
6. Rearrange: $2a^2c = ab^2 + bc^2$, i.e. $2(ca^2) = ab^2 + bc^2$.
7. This says the middle quantity ca^2 is the average of bc^2 and ab^2 , so bc^2, ca^2, ab^2 are in A.P.

Ans. bc^2, ca^2, ab^2 are in A.P. Hence proved.

Part 2: Relations Between the Roots

Q6. If α and β are the roots of $x^2 + x + 1 = 0$, find (i) $\alpha^2 + \beta^2$, (ii) $\alpha^3 + \beta^3$, (iii) $\alpha/\beta + \beta/\alpha$. Also show that $\alpha^3 = 1$.

Solution:

1. $\alpha + \beta = -1$ and $\alpha\beta = 1$.
2. **(i)** $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = 1 - 2 = -1$.

3. (ii) $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta) = -1 - 3(1)(-1) = -1 + 3 = 2.$

4. (iii) $\alpha/\beta + \beta/\alpha = (\alpha^2 + \beta^2)/\alpha\beta = -1/1 = -1.$

5. **To show $\alpha^3 = 1$:** α satisfies $\alpha^2 + \alpha + 1 = 0$. Multiply by $(\alpha - 1)$: $(\alpha - 1)(\alpha^2 + \alpha + 1) = \alpha^3 - 1 = 0.$

6. So $\alpha^3 = 1$. In the same way $\beta^3 = 1$, which agrees with (ii): $\alpha^3 + \beta^3 = 1 + 1 = 2 \checkmark$

Ans. (i) -1 (ii) 2 (iii) -1; $\alpha^3 = 1$.

Q7. If α and β are the roots of $x^2 - 4x + 1 = 0$, form the quadratic equation whose roots are (i) α^3 and β^3 , (ii) $\alpha - 1$ and $\beta - 1$, (iii) $2\alpha + 1$ and $2\beta + 1$.

Solution:

1. $\alpha + \beta = 4$ and $\alpha\beta = 1$.

2. (i) Sum = $\alpha^3 + \beta^3 = 64 - 3(1)(4) = 52$. Product = $(\alpha\beta)^3 = 1$. Equation: $x^2 - 52x + 1 = 0$.

3. (ii) Sum = $(\alpha + \beta) - 2 = 2$. Product = $\alpha\beta - (\alpha + \beta) + 1 = 1 - 4 + 1 = -2$. Equation: $x^2 - 2x - 2 = 0$.

4. (iii) Sum = $2(\alpha + \beta) + 2 = 10$. Product = $4\alpha\beta + 2(\alpha + \beta) + 1 = 4 + 8 + 1 = 13$. Equation: $x^2 - 10x + 13 = 0$.

5. Check (ii): the roots of the given equation are $2 \pm \sqrt{3}$, so the new roots are $1 \pm \sqrt{3}$, whose sum is 2 and product is -2 $\checkmark$

Ans. (i) $x^2 - 52x + 1 = 0$ (ii) $x^2 - 2x - 2 = 0$ (iii) $x^2 - 10x + 13 = 0$.

Q8. (a) The difference of the roots of $x^2 - 7x + k = 0$ is 3. Find k. (b) If the roots of $x^2 + px + q = 0$ differ by 1, prove that $p^2 = 4q + 1$.

Solution:

1. (a) Sum of roots = 7 and product = k. Use $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$.

2. Given $\alpha - \beta = 3$: $9 = 49 - 4k$, so $4k = 40$, so $k = 10$.

3. Check: $x^2 - 7x + 10 = (x - 2)(x - 5)$, roots 2 and 5, difference 3 $\checkmark$

4. (b) Sum of roots = $-p$ and product = q . Given $\alpha - \beta = 1$.

5. $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$, so $1 = p^2 - 4q$.

6. Hence $p^2 = 4q + 1$.

7. Check with $x^2 - 5x + 6 = 0$ (roots 2 and 3, difference 1): $p^2 = 25$ and $4q + 1 = 25 \checkmark$

Ans. (a) $k = 10$ (b) $p^2 = 4q + 1$. Hence proved.

Q9. Prove that if $a_1x^2 + b_1x + c_1 = 0$ and $a_2x^2 + b_2x + c_2 = 0$ have a common root, then $(c_1a_2 - c_2a_1)^2 = (a_1b_2 - a_2b_1)(b_1c_2 - b_2c_1)$.

Solution:

1. Let x be the common root. Then $a_1x^2 + b_1x + c_1 = 0 \dots$ (i) and $a_2x^2 + b_2x + c_2 = 0 \dots$ (ii)

2. Treat x^2 , x and 1 as three quantities and solve (i) and (ii) by cross-multiplication:

3. $x^2/(b_1c_2 - b_2c_1) = x/(c_1a_2 - c_2a_1) = 1/(a_1b_2 - a_2b_1).$

4. From the last two ratios: $x = (c_1a_2 - c_2a_1)/(a_1b_2 - a_2b_1)$.
5. From the first and last ratios: $x^2 = (b_1c_2 - b_2c_1)/(a_1b_2 - a_2b_1)$.
6. Square the value of x and equate it to the value of x^2 : $(c_1a_2 - c_2a_1)^2/(a_1b_2 - a_2b_1)^2 = (b_1c_2 - b_2c_1)/(a_1b_2 - a_2b_1)$.
7. Multiply by $(a_1b_2 - a_2b_1)^2$: $(c_1a_2 - c_2a_1)^2 = (a_1b_2 - a_2b_1)(b_1c_2 - b_2c_1)$.
8. Check with $x^2 - 3x + 2 = 0$ and $x^2 - 4x + 3 = 0$ (common root 1): $(2 - 3)^2 = 1$ and $(-4 + 3)(-9 + 8) = (-1)(-1) = 1 \checkmark$

Ans. Condition for a common root proved.

Part 3: Special Equations Reducible to Quadratic Form

Q10. Solve $(x + 1)(x + 2)(x + 3)(x + 4) = 24$.

Solution:

1. Pair the factors so that the x -terms match: $(x + 1)(x + 4) = x^2 + 5x + 4$ and $(x + 2)(x + 3) = x^2 + 5x + 6$.
2. Put $y = x^2 + 5x$. The equation becomes $(y + 4)(y + 6) = 24$.
3. Expand: $y^2 + 10y + 24 = 24$, so $y^2 + 10y = 0$, so $y(y + 10) = 0$, so $y = 0$ or $y = -10$.
4. Case $y = 0$: $x^2 + 5x = 0$, so $x(x + 5) = 0$, so $x = 0$ or $x = -5$.
5. Case $y = -10$: $x^2 + 5x + 10 = 0$. $D = 25 - 40 = -15 < 0$, so $x = (-5 \pm i\sqrt{15})/2$.
6. Check $x = 0$: $1 \times 2 \times 3 \times 4 = 24 \checkmark$. Check $x = -5$: $(-4)(-3)(-2)(-1) = 24 \checkmark$

Ans. $x = 0$, $x = -5$ (real roots) and $x = (-5 \pm i\sqrt{15})/2$ (complex roots).

Q11. Solve $x^4 - 10x^3 + 26x^2 - 10x + 1 = 0$.

Solution:

1. The coefficients read the same forwards and backwards (1, -10, 26, -10, 1), so divide by x^2 .
2. $x^2 - 10x + 26 - 10/x + 1/x^2 = 0$, i.e. $(x^2 + 1/x^2) - 10(x + 1/x) + 26 = 0$.
3. Put $y = x + 1/x$, so $x^2 + 1/x^2 = y^2 - 2$.
4. $y^2 - 2 - 10y + 26 = 0$, so $y^2 - 10y + 24 = 0$, so $(y - 4)(y - 6) = 0$, so $y = 4$ or $y = 6$.
5. Case $y = 4$: $x + 1/x = 4$, so $x^2 - 4x + 1 = 0$, so $x = 2 \pm \sqrt{3}$.
6. Case $y = 6$: $x + 1/x = 6$, so $x^2 - 6x + 1 = 0$, so $x = 3 \pm 2\sqrt{2}$.
7. Check $x = 2 + \sqrt{3}$: $1/x = 2 - \sqrt{3}$, so $x + 1/x = 4 \checkmark$

Ans. $x = 2 + \sqrt{3}$, $2 - \sqrt{3}$, $3 + 2\sqrt{2}$, $3 - 2\sqrt{2}$.

Q12. Solve (a) $(\log_{10} x)^2 - 3 \log_{10} x + 2 = 0$ (b) $(\log_2 x)^2 - 5 \log_2 x + 6 = 0$.

Solution:

1. (a) Put $y = \log_{10} x$. Then $y^2 - 3y + 2 = 0$, so $(y - 1)(y - 2) = 0$, so $y = 1$ or $y = 2$.
2. $\log_{10} x = 1$ gives $x = 10$; $\log_{10} x = 2$ gives $x = 100$.
3. Check $x = 100$: $2^2 - 3(2) + 2 = 0 \checkmark$

4. **(b)** Put $y = \log_2 x$. Then $y^2 - 5y + 6 = 0$, so $(y - 2)(y - 3) = 0$, so $y = 2$ or $y = 3$.
5. $\log_2 x = 2$ gives $x = 4$; $\log_2 x = 3$ gives $x = 8$.
6. Check $x = 8$: $3^2 - 5(3) + 6 = 0$ ✓

Ans. (a) $x = 10$ or $x = 100$ (b) $x = 4$ or $x = 8$.

Q13. Solve (a) $x^2 - 5|x| + 6 = 0$ (b) $x^2 - |x| - 6 = 0$ (c) $2\cos^2\theta + \cos\theta - 1 = 0$ for $0^\circ \leq \theta \leq 360^\circ$.

Solution:

1. **(a)** Since $x^2 = |x|^2$, put $y = |x|$: $y^2 - 5y + 6 = 0$, so $(y - 2)(y - 3) = 0$, so $|x| = 2$ or $|x| = 3$.
2. So $x = \pm 2$ or $x = \pm 3$.
3. **(b)** Put $y = |x|$: $y^2 - y - 6 = 0$, so $(y - 3)(y + 2) = 0$, so $y = 3$ or $y = -2$.
4. Since $|x|$ cannot be negative, $y = -2$ is rejected. So $|x| = 3$, i.e. $x = \pm 3$.
5. **(c)** Put $c = \cos\theta$: $2c^2 + c - 1 = 0$, so $(2c - 1)(c + 1) = 0$, so $c = 1/2$ or $c = -1$.
6. $\cos\theta = 1/2$ gives $\theta = 60^\circ$ or 300° . $\cos\theta = -1$ gives $\theta = 180^\circ$.

Ans. (a) $x = \pm 2, \pm 3$ (b) $x = \pm 3$ (c) $\theta = 60^\circ, 180^\circ, 300^\circ$.

Part 4: Applications

Q14. The sum of the ages of two friends is 20 years. Four years ago, the product of their ages was 32. Find their present ages.

Solution:

1. Let the age of one friend = x years. Then the other friend's age = $(20 - x)$ years.
2. Four years ago their ages were $(x - 4)$ and $(16 - x)$.
3. Given: $(x - 4)(16 - x) = 32$.
4. Expand: $16x - x^2 - 64 + 4x = 32$, so $-x^2 + 20x - 96 = 0$, so $x^2 - 20x + 96 = 0$.
5. $D = 400 - 384 = 16$, so $x = (20 \pm 4)/2 = 12$ or 8 .
6. So the ages are 12 and 8 years (the two roots give the same pair).
7. Check: four years ago their ages were 8 and 4, and $8 \times 4 = 32$ ✓

Ans. The present ages are 12 years and 8 years.

Q15. A trader buys a number of articles for Rs 1200. Had he bought 10 more articles for the same amount, each article would have cost Rs 10 less. Find the number of articles bought and the cost of each.

Solution:

1. Let the number of articles = x . Cost of each article = Rs $1200/x$.
2. If the number is $(x + 10)$, the cost of each = Rs $1200/(x + 10)$.
3. Given: $1200/x - 1200/(x + 10) = 10$.
4. Combine: $1200 \times 10 / [x(x + 10)] = 10$, so $x(x + 10) = 1200$.
5. $x^2 + 10x - 1200 = 0$. $D = 100 + 4800 = 4900 = 70^2$.

6. $x = (-10 \pm 70)/2 = 30$ or -40 . The number cannot be negative, so $x = 30$.
7. Cost of each article = $1200/30 = \text{Rs } 40$.
8. Check: with 40 articles, each costs $1200/40 = \text{Rs } 30$, which is Rs 10 less ✓

Ans. 30 articles were bought, at Rs 40 each.

Q16. The perimeter of a rectangle is 34 cm and its diagonal is 13 cm. Find its length and breadth.

Solution:

1. Let the length = l and the breadth = b .
2. Perimeter: $2(l + b) = 34$, so $l + b = 17 \dots (i)$
3. Diagonal: $l^2 + b^2 = 13^2 = 169 \dots (ii)$
4. Use $(l + b)^2 = l^2 + b^2 + 2lb$: $289 = 169 + 2lb$, so $lb = 60$.
5. So l and b are the roots of $t^2 - (l + b)t + lb = 0$, i.e. $t^2 - 17t + 60 = 0$.
6. $(t - 12)(t - 5) = 0$, so $t = 12$ or $t = 5$.
7. Check: $12^2 + 5^2 = 144 + 25 = 169 \checkmark$ and $2(12 + 5) = 34 \checkmark$

Ans. Length = 12 cm and breadth = 5 cm.

Q17. Two workers A and B together can complete a job in 12 days. A alone takes 10 days more than B alone. Find the time each takes to complete the job alone.

Solution:

1. Let B alone take x days. Then A alone takes $(x + 10)$ days.
2. In one day, B does $1/x$ of the job and A does $1/(x + 10)$. Together: $1/x + 1/(x + 10) = 1/12$.
3. Combine: $(2x + 10)/[x(x + 10)] = 1/12$, so $12(2x + 10) = x(x + 10)$.
4. $24x + 120 = x^2 + 10x$, so $x^2 - 14x - 120 = 0$.
5. $D = 196 + 480 = 676 = 26^2$, so $x = (14 \pm 26)/2 = 20$ or -6 .
6. Time cannot be negative, so $x = 20$ and $x + 10 = 30$.
7. Check: $1/20 + 1/30 = 3/60 + 2/60 = 5/60 = 1/12 \checkmark$

Ans. B takes 20 days and A takes 30 days.