

# MATHEMATICS 1

## Chapter: Vector – Scalar and Dot Product

### Long Answer Questions with Solutions

These notes are prepared for engineering students. They contain the important long-answer questions from the chapter, each with step-by-step working and the final answer.

**Q1. Find the angle between the vectors  $\mathbf{a} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$  and  $\mathbf{b} = \mathbf{i} + \mathbf{j}$ .**

**Step 1:**  $\mathbf{a} \cdot \mathbf{b} = (2)(1) + (1)(1) + (1)(0) = 3$

**Step 2:**  $|\mathbf{a}| = \sqrt{4 + 1 + 1} = \sqrt{6}$ ,  $|\mathbf{b}| = \sqrt{1 + 1} = \sqrt{2}$

**Step 3:**  $\cos \theta = (\mathbf{a} \cdot \mathbf{b}) / (|\mathbf{a}||\mathbf{b}|) = 3 / (\sqrt{6} \times \sqrt{2}) = 3 / \sqrt{12} = 3 / (2\sqrt{3}) = \sqrt{3} / 2$

**Answer:**  $\theta = 30^\circ$

**Q2. Show that the vectors  $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$ ,  $\mathbf{b} = \mathbf{i} - 3\mathbf{j} - 5\mathbf{k}$  and  $\mathbf{c} = 3\mathbf{i} - 4\mathbf{j} - 4\mathbf{k}$  form a right-angled triangle.**

**Step 1:**  $\mathbf{a} + \mathbf{b} = (2 + 1)\mathbf{i} + (-1 - 3)\mathbf{j} + (1 - 5)\mathbf{k} = 3\mathbf{i} - 4\mathbf{j} - 4\mathbf{k} = \mathbf{c}$ . Since  $\mathbf{a} + \mathbf{b} = \mathbf{c}$ , the three vectors form the sides of a triangle.

**Step 2:**  $\mathbf{a} \cdot \mathbf{b} = (2)(1) + (-1)(-3) + (1)(-5) = 2 + 3 - 5 = 0$ , so  $\mathbf{a}$  is perpendicular to  $\mathbf{b}$ .

**Step 3 (check):**  $|\mathbf{a}|^2 = 4 + 1 + 1 = 6$ ,  $|\mathbf{b}|^2 = 1 + 9 + 25 = 35$ ,  $|\mathbf{c}|^2 = 9 + 16 + 16 = 41$ . Since  $6 + 35 = 41$ ,  $|\mathbf{a}|^2 + |\mathbf{b}|^2 = |\mathbf{c}|^2$ .

**Answer:** The triangle is right-angled; the right angle is between the sides  $\mathbf{a}$  and  $\mathbf{b}$ , and  $\mathbf{c}$  is the hypotenuse.

**Q3. Find the value of  $\lambda$  for which the vector  $\mathbf{a} + \lambda\mathbf{b}$  is perpendicular to  $\mathbf{c}$ , where  $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ ,  $\mathbf{b} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$  and  $\mathbf{c} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ .**

**Step 1:**  $\mathbf{a} + \lambda\mathbf{b} = (1 + 2\lambda)\mathbf{i} + (2 - \lambda)\mathbf{j} + (3 + \lambda)\mathbf{k}$

**Step 2:** For perpendicular vectors,  $(\mathbf{a} + \lambda\mathbf{b}) \cdot \mathbf{c} = 0$ :

$$(1 + 2\lambda)(1) + (2 - \lambda)(1) + (3 + \lambda)(1) = 0$$

**Step 3:**  $6 + 2\lambda = 0 \Rightarrow \lambda = -3$

**Check:** For  $\lambda = -3$ ,  $\mathbf{a} + \lambda\mathbf{b} = -5\mathbf{i} + 5\mathbf{j} + 0\mathbf{k}$ , and its dot product with  $\mathbf{c}$  is  $-5 + 5 + 0 = 0$ .

**Answer:**  $\lambda = -3$

**Q4. Find the scalar projection and the vector projection of  $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$  on  $\mathbf{b} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ .**

**Step 1:**  $\mathbf{a} \cdot \mathbf{b} = (2)(1) + (-1)(-2) + (1)(2) = 2 + 2 + 2 = 6$

**Step 2:**  $|\mathbf{b}| = \sqrt{1 + 4 + 4} = 3$

**Step 3:** Scalar projection =  $(\mathbf{a} \cdot \mathbf{b}) / |\mathbf{b}| = 6/3 = 2$

**Step 4:** Vector projection =  $[(\mathbf{a} \cdot \mathbf{b}) / |\mathbf{b}|^2] \mathbf{b} = (6/9)(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}) = (2/3)(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$

**Answer:** Scalar projection = 2; vector projection =  $(2/3)\mathbf{i} - (4/3)\mathbf{j} + (4/3)\mathbf{k}$

**Q5. A force  $\mathbf{F} = 3\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$  newtons displaces a particle from the point A(1, 2, 3) to the point B(5, 4, 1). Find the work done by the force.**

**Step 1:** Displacement  $\mathbf{d} = \mathbf{AB} = (5 - 1)\mathbf{i} + (4 - 2)\mathbf{j} + (1 - 3)\mathbf{k} = 4\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$

**Step 2:** Work done  $W = \mathbf{F} \cdot \mathbf{d}$

**Step 3:**  $W = (3)(4) + (2)(2) + (-4)(-2) = 12 + 4 + 8 = 24$

**Answer:**  $W = 24$  joules (taking displacement in metres)

**Q6. Prove that  $|a + b|^2 + |a - b|^2 = 2(|a|^2 + |b|^2)$ . (Parallelogram law)**

**Step 1:**  $|a + b|^2 = (a + b) \cdot (a + b) = a \cdot a + 2a \cdot b + b \cdot b = |a|^2 + |b|^2 + 2a \cdot b$

**Step 2:**  $|a - b|^2 = (a - b) \cdot (a - b) = |a|^2 + |b|^2 - 2a \cdot b$

**Step 3:** Adding the two results, the terms  $\pm 2a \cdot b$  cancel:

$$|a + b|^2 + |a - b|^2 = 2|a|^2 + 2|b|^2 = 2(|a|^2 + |b|^2)$$

**Answer:** Hence proved. In a parallelogram, the sum of the squares of the diagonals equals the sum of the squares of the four sides.

**Q7. Using vectors, prove that the diagonals of a rhombus are perpendicular to each other.**

**Step 1:** Let OABC be a rhombus with  $OA = a$  and  $OC = b$ . All sides are equal, so  $|a| = |b|$ .

**Step 2:** Diagonal  $OB = OA + AB = a + b$  (since  $AB = OC = b$ ). Diagonal  $AC = OC - OA = b - a$ .

**Step 3:**  $(a + b) \cdot (b - a) = a \cdot b - a \cdot a + b \cdot b - b \cdot a = |b|^2 - |a|^2$

**Step 4:** Since  $|a| = |b|$ , this equals 0.

**Answer:**  $OB \cdot AC = 0$ , so the diagonals  $OB$  and  $AC$  are perpendicular. Hence proved.

**Q8. If  $a + b + c = 0$ ,  $|a| = 3$ ,  $|b| = 5$  and  $|c| = 7$ , find the angle between  $a$  and  $b$ .**

**Step 1:**  $a + b = -c$ , so  $|a + b|^2 = |c|^2$

**Step 2:**  $|a|^2 + |b|^2 + 2a \cdot b = |c|^2 \Rightarrow 9 + 25 + 2a \cdot b = 49 \Rightarrow a \cdot b = 15/2$

**Step 3:**  $\cos \theta = (a \cdot b) / (|a||b|) = (15/2) / (3 \times 5) = 1/2$

**Answer:**  $\theta = 60^\circ$

**Q9. If  $|a| = 2$ ,  $|b| = 3$  and  $a \cdot b = 3$ , find  $|a + b|$ ,  $|a - b|$  and the angle between  $a$  and  $b$ .**

**Step 1:**  $|a + b|^2 = |a|^2 + |b|^2 + 2a \cdot b = 4 + 9 + 6 = 19$

**Step 2:**  $|a - b|^2 = |a|^2 + |b|^2 - 2a \cdot b = 4 + 9 - 6 = 7$

**Step 3:**  $\cos \theta = 3 / (2 \times 3) = 1/2$

**Answer:**  $|a + b| = \sqrt{19}$ ,  $|a - b| = \sqrt{7}$ ,  $\theta = 60^\circ$

**Q10. Using vectors, prove the cosine rule  $c^2 = a^2 + b^2 - 2ab \cos C$  for a triangle ABC.**

**Step 1:** Let  $BC = p$ ,  $CA = q$  and  $AB = r$ , with  $|p| = a$ ,  $|q| = b$  and  $|r| = c$ . Going round the triangle,  $p + q + r = 0$ , so  $r = -(p + q)$ .

**Step 2:**  $|r|^2 = |p + q|^2 = |p|^2 + |q|^2 + 2p \cdot q$

**Step 3:** The vectors  $p$  (B to C) and  $q$  (C to A) follow each other, so the angle between them is  $180^\circ - C$ . Therefore  $p \cdot q = ab \cos(180^\circ - C) = -ab \cos C$ .

**Step 4:** Substituting:  $c^2 = a^2 + b^2 - 2ab \cos C$ .

**Answer:** Hence proved.

**Q11. Find the angle between a diagonal of a cube and one of its edges.**

**Step 1:** Take a cube of side 1 with one vertex at the origin and three edges along the axes. The diagonal from the origin is  $\mathbf{d} = \mathbf{i} + \mathbf{j} + \mathbf{k}$  and one edge is  $\mathbf{e} = \mathbf{i}$ .

**Step 2:**  $\mathbf{d} \cdot \mathbf{e} = 1$ ,  $|\mathbf{d}| = \sqrt{3}$ ,  $|\mathbf{e}| = 1$

**Step 3:**  $\cos \theta = 1 / \sqrt{3}$

**Answer:**  $\theta = \cos^{-1}(1/\sqrt{3}) \approx 54.74^\circ$

**Q12. Find the direction cosines and the direction angles of the vector  $\mathbf{a} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ .**

**Step 1:**  $|\mathbf{a}| = \sqrt{4 + 4 + 1} = 3$

**Step 2:**  $l = 2/3$ ,  $m = -2/3$ ,  $n = 1/3$

**Step 3 (check):**  $l^2 + m^2 + n^2 = 4/9 + 4/9 + 1/9 = 1$

**Step 4:**  $\alpha = \cos^{-1}(2/3) \approx 48.19^\circ$ ,  $\beta = \cos^{-1}(-2/3) \approx 131.81^\circ$ ,  $\gamma = \cos^{-1}(1/3) \approx 70.53^\circ$

**Answer:** Direction cosines:  $2/3, -2/3, 1/3$

**Q13. Resolve the vector  $\mathbf{a} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$  into components parallel and perpendicular to  $\mathbf{b} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ .**

**Step 1:**  $\mathbf{a} \cdot \mathbf{b} = (3)(1) + (-1)(2) + (2)(-2) = 3 - 2 - 4 = -3$

**Step 2:**  $|\mathbf{b}|^2 = 1 + 4 + 4 = 9$

**Step 3:** Component parallel to  $\mathbf{b} = [(\mathbf{a} \cdot \mathbf{b}) / |\mathbf{b}|^2] \mathbf{b} = (-3/9)(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}) = -(1/3)\mathbf{i} - (2/3)\mathbf{j} + (2/3)\mathbf{k}$

**Step 4:** Component perpendicular to  $\mathbf{b} = \mathbf{a} - (\text{parallel component}) = (3 + 1/3)\mathbf{i} + (-1 + 2/3)\mathbf{j} + (2 - 2/3)\mathbf{k} = (10/3)\mathbf{i} - (1/3)\mathbf{j} + (4/3)\mathbf{k}$

**Check:** The perpendicular component dotted with  $\mathbf{b}$  gives  $10/3 - 2/3 - 8/3 = 0$ .

**Answer:** Parallel:  $-(1/3)\mathbf{i} - (2/3)\mathbf{j} + (2/3)\mathbf{k}$ ; perpendicular:  $(10/3)\mathbf{i} - (1/3)\mathbf{j} + (4/3)\mathbf{k}$

**Q14. Find the angle between the diagonals of the parallelogram whose adjacent sides are  $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$  and  $\mathbf{b} = \mathbf{i} - \mathbf{k}$ .**

**Step 1:** The diagonals are  $\mathbf{a} + \mathbf{b}$  and  $\mathbf{a} - \mathbf{b}$ .

**Step 2:**  $\mathbf{a} + \mathbf{b} = 2\mathbf{i} + 2\mathbf{j} + 0\mathbf{k}$  and  $\mathbf{a} - \mathbf{b} = 0\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$

**Step 3:**  $(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = (2)(0) + (2)(2) + (0)(2) = 4$

**Step 4:**  $|\mathbf{a} + \mathbf{b}| = \sqrt{8}$  and  $|\mathbf{a} - \mathbf{b}| = \sqrt{8}$ , so  $\cos \theta = 4 / (\sqrt{8} \times \sqrt{8}) = 4/8 = 1/2$

**Answer:**  $\theta = 60^\circ$

**Q15. If  $\mathbf{a}$  and  $\mathbf{b}$  are unit vectors and  $\theta$  is the angle between them, prove that  $|\mathbf{a} - \mathbf{b}| = 2 \sin(\theta/2)$ .**

**Step 1:**  $|\mathbf{a}| = |\mathbf{b}| = 1$ , so  $\mathbf{a} \cdot \mathbf{b} = \cos \theta$ .

**Step 2:**  $|\mathbf{a} - \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\mathbf{a} \cdot \mathbf{b} = 1 + 1 - 2 \cos \theta = 2(1 - \cos \theta)$

**Step 3:** Using  $1 - \cos \theta = 2 \sin^2(\theta/2)$ , we get  $|\mathbf{a} - \mathbf{b}|^2 = 4 \sin^2(\theta/2)$ .

**Step 4:** Taking the square root ( $\sin(\theta/2) \geq 0$  because  $0^\circ \leq \theta \leq 180^\circ$ ):  $|\mathbf{a} - \mathbf{b}| = 2 \sin(\theta/2)$ .

**Answer:** Hence proved.

**Q16. Using vectors, prove that the angle in a semicircle is a right angle.**

**Step 1:** Let O be the centre of a circle of radius r, AB a diameter and P any point on the circle. Taking O as the origin,  $OA = \mathbf{a}$ ,  $OB = -\mathbf{a}$  and  $OP = \mathbf{p}$ , with  $|\mathbf{a}| = |\mathbf{p}| = r$ .

**Step 2:**  $PA = OA - OP = \mathbf{a} - \mathbf{p}$  and  $PB = OB - OP = -\mathbf{a} - \mathbf{p}$

**Step 3:**  $PA \cdot PB = (\mathbf{a} - \mathbf{p}) \cdot (-\mathbf{a} - \mathbf{p}) = -\mathbf{a} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{p} + \mathbf{p} \cdot \mathbf{a} + \mathbf{p} \cdot \mathbf{p} = -|\mathbf{a}|^2 + |\mathbf{p}|^2$

**Step 4:** Since  $|\mathbf{a}| = |\mathbf{p}| = r$ ,  $PA \cdot PB = -r^2 + r^2 = 0$ .

**Answer:** PA is perpendicular to PB, so angle APB =  $90^\circ$ . Hence proved.

**Q17. Find a unit vector perpendicular to both  $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$  and  $\mathbf{b} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ .**

**Step 1:** Let the required vector be  $\mathbf{c} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ . Perpendicular to  $\mathbf{a}$  means  $\mathbf{c} \cdot \mathbf{a} = 0$ , and perpendicular to  $\mathbf{b}$  means  $\mathbf{c} \cdot \mathbf{b} = 0$ .

**Step 2:**  $x + y + z = 0 \dots(1)$  and  $x + 2y + 3z = 0 \dots(2)$

**Step 3:** Subtracting (1) from (2):  $y + 2z = 0$ , so  $y = -2z$ . Putting this in (1):  $x - 2z + z = 0$ , so  $x = z$ .

**Step 4:** Taking  $z = 1$ ,  $\mathbf{c} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$  and  $|\mathbf{c}| = \sqrt{(1 + 4 + 1)} = \sqrt{6}$ .

**Step 5:** Unit vector =  $\mathbf{c} / |\mathbf{c}|$

**Answer:**  $\pm(\mathbf{i} - 2\mathbf{j} + \mathbf{k}) / \sqrt{6}$

**Q18. If  $|\mathbf{a} + \mathbf{b}| = |\mathbf{a} - \mathbf{b}|$ , prove that  $\mathbf{a}$  and  $\mathbf{b}$  are perpendicular.**

**Step 1:** Square both sides:  $|\mathbf{a} + \mathbf{b}|^2 = |\mathbf{a} - \mathbf{b}|^2$

**Step 2:**  $|\mathbf{a}|^2 + |\mathbf{b}|^2 + 2\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\mathbf{a} \cdot \mathbf{b}$

**Step 3:**  $4\mathbf{a} \cdot \mathbf{b} = 0 \Rightarrow \mathbf{a} \cdot \mathbf{b} = 0$

**Answer:** Since  $\mathbf{a} \cdot \mathbf{b} = 0$ , the non-zero vectors  $\mathbf{a}$  and  $\mathbf{b}$  are perpendicular. Hence proved.  
(Geometrically: the diagonals of a parallelogram are equal only when it is a rectangle.)

**Q19. Find the values of  $\lambda$  for which the angle between  $\mathbf{a} = 2\lambda^2\mathbf{i} + 4\lambda\mathbf{j} + \mathbf{k}$  and  $\mathbf{b} = 7\mathbf{i} - 2\mathbf{j} + \lambda\mathbf{k}$  is obtuse.**

**Step 1:** For an obtuse angle,  $\cos \theta < 0$ , so  $\mathbf{a} \cdot \mathbf{b} < 0$ .

**Step 2:**  $\mathbf{a} \cdot \mathbf{b} = (2\lambda^2)(7) + (4\lambda)(-2) + (1)(\lambda) = 14\lambda^2 - 8\lambda + \lambda = 14\lambda^2 - 7\lambda = 7\lambda(2\lambda - 1)$

**Step 3:**  $7\lambda(2\lambda - 1) < 0$  when  $\lambda$  and  $(2\lambda - 1)$  have opposite signs, i.e.  $0 < \lambda < 1/2$ .

**Step 4:** The vectors cannot point in exactly opposite directions here. That would need  $\mathbf{a} = m\mathbf{b}$  with  $m < 0$ , which leads to  $-2\lambda^2 = 1$ , and this has no real solution. So the whole interval is valid.

**Answer:**  $0 < \lambda < 1/2$

**Q20. A force of magnitude 6 N acts in the direction of  $\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$  and displaces a particle from A(0, 1, 2) to B(3, 2, 4). Find the work done.**

**Step 1:** Unit vector along the force =  $(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) / \sqrt{(1 + 4 + 4)} = (\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) / 3$

**Step 2:**  $\mathbf{F} = 6 \times (\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) / 3 = 2\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}$

**Step 3:**  $\mathbf{d} = AB = (3 - 0)\mathbf{i} + (2 - 1)\mathbf{j} + (4 - 2)\mathbf{k} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$

**Step 4:**  $W = \mathbf{F} \cdot \mathbf{d} = (2)(3) + (4)(1) + (4)(2) = 6 + 4 + 8 = 18$

**Answer:**  $W = 18$  joules

**Q21. Find the angles of the triangle whose vertices are A(1, 2, 3), B(2, 3, 1) and C(3, 1, 2).**

**Step 1:**  $AB = i + j - 2k$ ,  $BC = i - 2j + k$ ,  $CA = -2i + j + k$

**Step 2:**  $|AB| = |BC| = |CA| = \sqrt{6}$ , so all three sides are equal.

**Step 3:** The angle at A is between AB and AC, where  $AC = 2i - j - k$ .  $AB \cdot AC = 2 - 1 + 2 = 3$ , so  $\cos A = 3 / (\sqrt{6} \times \sqrt{6}) = 1/2$ , giving  $A = 60^\circ$ .

**Step 4:** The triangle is equilateral, so  $B = C = 60^\circ$  as well (the same steps at B and C confirm this).

**Answer:**  $A = B = C = 60^\circ$  (equilateral triangle)