

MATHEMATICS 1

Chapter: Vector – Scalar and Dot Product

Long Answer Questions with Solutions (Set 2)

This is a second set of important long-answer questions from the chapter for engineering students. Each question has step-by-step working and the final answer.

Q1. Find the unit vector in the direction of $\mathbf{a} + \mathbf{b}$, where $\mathbf{a} = 2\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}$ and $\mathbf{b} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$. Also find $\mathbf{a} \cdot \mathbf{b}$.

Step 1: $\mathbf{a} + \mathbf{b} = (2 + 1)\mathbf{i} + (4 + 2)\mathbf{j} + (-5 + 3)\mathbf{k} = 3\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}$

Step 2: $|\mathbf{a} + \mathbf{b}| = \sqrt{(9 + 36 + 4)} = \sqrt{49} = 7$

Step 3: Unit vector $= (\mathbf{a} + \mathbf{b}) / |\mathbf{a} + \mathbf{b}| = (3\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}) / 7$

Step 4: $\mathbf{a} \cdot \mathbf{b} = (2)(1) + (4)(2) + (-5)(3) = 2 + 8 - 15 = -5$

Answer: Unit vector $= (3/7)\mathbf{i} + (6/7)\mathbf{j} - (2/7)\mathbf{k}$; $\mathbf{a} \cdot \mathbf{b} = -5$

Q2. Derive the component form of the dot product: if $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, show that $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$.

Step 1: $\mathbf{a} \cdot \mathbf{b} = (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \cdot (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k})$

Step 2: Expanding with the distributive law gives nine terms: $a_1b_1(\mathbf{i} \cdot \mathbf{i}) + a_1b_2(\mathbf{i} \cdot \mathbf{j}) + a_1b_3(\mathbf{i} \cdot \mathbf{k}) + a_2b_1(\mathbf{j} \cdot \mathbf{i}) + a_2b_2(\mathbf{j} \cdot \mathbf{j}) + a_2b_3(\mathbf{j} \cdot \mathbf{k}) + a_3b_1(\mathbf{k} \cdot \mathbf{i}) + a_3b_2(\mathbf{k} \cdot \mathbf{j}) + a_3b_3(\mathbf{k} \cdot \mathbf{k})$

Step 3: Use $\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1$ and $\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0$ (and the same with the order reversed). Every mixed term vanishes.

Step 4: Only three terms remain. Equating with $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$ also gives $\cos \theta = (a_1b_1 + a_2b_2 + a_3b_3) / (|\mathbf{a}||\mathbf{b}|)$.

Answer: $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$. Hence proved.

Q3. Find a unit vector that lies in the xy-plane and is perpendicular to $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j} + \mathbf{k}$.

Step 1: A vector in the xy-plane has no $\mathbf{k}$ component, so let $\mathbf{c} = x\mathbf{i} + y\mathbf{j}$.

Step 2: Unit vector: $x^2 + y^2 = 1$. Perpendicular to $\mathbf{a}$: $\mathbf{c} \cdot \mathbf{a} = 3x + 4y = 0$, so $y = -3x/4$.

Step 3: Substituting: $x^2 + 9x^2/16 = 1 \Rightarrow 25x^2/16 = 1 \Rightarrow x^2 = 16/25 \Rightarrow x = \pm 4/5$, and then $y = \mp 3/5$.

Check: For $x = 4/5$ and $y = -3/5$, $\mathbf{c} \cdot \mathbf{a} = 12/5 - 12/5 = 0$.

Answer: $\pm(4\mathbf{i} - 3\mathbf{j}) / 5$

Q4. Find the vector $\mathbf{a}$ if $\mathbf{a} \cdot (\mathbf{i} + \mathbf{j}) = 5$, $\mathbf{a} \cdot (\mathbf{j} + \mathbf{k}) = 4$ and $\mathbf{a} \cdot (\mathbf{k} + \mathbf{i}) = 3$.

Step 1: Let $\mathbf{a} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. The three conditions give $x + y = 5$, $y + z = 4$, $z + x = 3$.

Step 2: Adding all three: $2(x + y + z) = 12$, so $x + y + z = 6$.

Step 3: $z = 6 - 5 = 1$, $x = 6 - 4 = 2$, $y = 6 - 3 = 3$.

Check: $y + z = 3 + 1 = 4$ and $z + x = 1 + 2 = 3$, as required.

Answer: $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$

Q5. Show that the vectors $(2\mathbf{i} - 2\mathbf{j} + \mathbf{k})/3$, $(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})/3$ and $(2\mathbf{i} + \mathbf{j} - 2\mathbf{k})/3$ are mutually perpendicular unit vectors.

Step 1: Name them $\mathbf{p} = (2\mathbf{i} - 2\mathbf{j} + \mathbf{k})/3$, $\mathbf{q} = (\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})/3$, $\mathbf{r} = (2\mathbf{i} + \mathbf{j} - 2\mathbf{k})/3$.

Step 2 (magnitudes): $|\mathbf{p}| = \sqrt{(4 + 4 + 1)}/3 = 1$, $|\mathbf{q}| = \sqrt{(1 + 4 + 4)}/3 = 1$, $|\mathbf{r}| = \sqrt{(4 + 1 + 4)}/3 = 1$

Step 3 (dot products): $\mathbf{p} \cdot \mathbf{q} = (2 - 4 + 2)/9 = 0$, $\mathbf{q} \cdot \mathbf{r} = (2 + 2 - 4)/9 = 0$, $\mathbf{p} \cdot \mathbf{r} = (4 - 2 - 2)/9 = 0$

Answer: Each vector has magnitude 1 and every pair has dot product 0, so they are mutually perpendicular unit vectors.

Q6. Find the projection of the vector AB on the vector CD, where A(0, 1, 2), B(3, 5, 2), C(1, 1, 1) and D(2, 3, 3).

Step 1: $\mathbf{AB} = (3 - 0)\mathbf{i} + (5 - 1)\mathbf{j} + (2 - 2)\mathbf{k} = 3\mathbf{i} + 4\mathbf{j}$

Step 2: $\mathbf{CD} = (2 - 1)\mathbf{i} + (3 - 1)\mathbf{j} + (3 - 1)\mathbf{k} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$

Step 3: $\mathbf{AB} \cdot \mathbf{CD} = (3)(1) + (4)(2) + (0)(2) = 11$, $|\mathbf{CD}| = \sqrt{(1 + 4 + 4)} = 3$

Step 4: Projection of AB on CD = $(\mathbf{AB} \cdot \mathbf{CD}) / |\mathbf{CD}| = 11/3$

Answer: $11/3 \approx 3.67$

Q7. Find the value of λ for which the angle between $\mathbf{a} = \mathbf{i} + \mathbf{j}$ and $\mathbf{b} = \lambda\mathbf{i} + \mathbf{j} + \mathbf{k}$ is 60° .

Step 1: $\mathbf{a} \cdot \mathbf{b} = \lambda + 1$, $|\mathbf{a}| = \sqrt{2}$, $|\mathbf{b}| = \sqrt{\lambda^2 + 2}$

Step 2: $\cos 60^\circ = 1/2 = (\lambda + 1) / (\sqrt{2} \cdot \sqrt{\lambda^2 + 2}) \Rightarrow 2(\lambda + 1) = \sqrt{2} \cdot \sqrt{\lambda^2 + 2}$

Step 3: Squaring: $4(\lambda + 1)^2 = 2(\lambda^2 + 2) \Rightarrow 2\lambda^2 + 4\lambda + 2 = \lambda^2 + 2 \Rightarrow \lambda^2 + 4\lambda = 0 \Rightarrow \lambda = 0$ or $\lambda = -4$

Step 4: Since $\cos 60^\circ$ is positive, $\mathbf{a} \cdot \mathbf{b} = \lambda + 1$ must be positive. So $\lambda = -4$ is rejected (it appeared because of squaring).

Check: For $\lambda = 0$, $\mathbf{b} = \mathbf{j} + \mathbf{k}$, $\mathbf{a} \cdot \mathbf{b} = 1$ and $\cos \theta = 1/(\sqrt{2} \cdot \sqrt{2}) = 1/2$.

Answer: $\lambda = 0$

Q8. Find the angle between two diagonals of a cube.

Step 1: Take a cube of side 1 with one vertex at the origin. One body diagonal goes from (0, 0, 0) to (1, 1, 1), so $\mathbf{d}_1 = \mathbf{i} + \mathbf{j} + \mathbf{k}$. Another goes from (1, 0, 0) to (0, 1, 1), so $\mathbf{d}_2 = -\mathbf{i} + \mathbf{j} + \mathbf{k}$.

Step 2: $\mathbf{d}_1 \cdot \mathbf{d}_2 = -1 + 1 + 1 = 1$, $|\mathbf{d}_1| = |\mathbf{d}_2| = \sqrt{3}$

Step 3: $\cos \theta = 1 / (\sqrt{3} \times \sqrt{3}) = 1/3$

Answer: $\theta = \cos^{-1}(1/3) \approx 70.53^\circ$

Q9. Find the angle between a diagonal of a cube and a diagonal of one of its faces (both starting from the same vertex).

Step 1: Take a cube of side 1 with one vertex at the origin. Body diagonal $\mathbf{d} = \mathbf{i} + \mathbf{j} + \mathbf{k}$; diagonal of the bottom face $\mathbf{f} = \mathbf{i} + \mathbf{j}$.

Step 2: $\mathbf{d} \cdot \mathbf{f} = 1 + 1 + 0 = 2$, $|\mathbf{d}| = \sqrt{3}$, $|\mathbf{f}| = \sqrt{2}$

Step 3: $\cos \theta = 2 / (\sqrt{3} \times \sqrt{2}) = 2/\sqrt{6} = \sqrt{6}/3 \approx 0.8165$

Answer: $\theta = \cos^{-1}(\sqrt{6}/3) \approx 35.26^\circ$

Q10. A vector makes angles of 45° and 60° with the positive x-axis and y-axis. Find the angle it makes with the z-axis.

Step 1: $l = \cos 45^\circ = 1/\sqrt{2}$, $m = \cos 60^\circ = 1/2$

Step 2: Use $l^2 + m^2 + n^2 = 1$: $1/2 + 1/4 + n^2 = 1 \Rightarrow n^2 = 1/4 \Rightarrow n = \pm 1/2$

Step 3: $\cos \gamma = \pm 1/2$, so $\gamma = 60^\circ$ or $\gamma = 120^\circ$.

Answer: $\gamma = 60^\circ$ or 120° (direction cosines: $1/\sqrt{2}$, $1/2$, $\pm 1/2$)

Q11. Find the vector of magnitude $3\sqrt{3}$ that makes equal angles with the three coordinate axes.

Step 1: Equal angles mean $l = m = n$. From $l^2 + m^2 + n^2 = 1$ we get $3l^2 = 1$, so $l = 1/\sqrt{3}$ (for angles with the positive axes).

Step 2: The unit vector is $(\mathbf{i} + \mathbf{j} + \mathbf{k})/\sqrt{3}$.

Step 3: Vector = $3\sqrt{3} \times (\mathbf{i} + \mathbf{j} + \mathbf{k})/\sqrt{3} = 3(\mathbf{i} + \mathbf{j} + \mathbf{k})$

Check: Its magnitude is $\sqrt{(9 + 9 + 9)} = \sqrt{27} = 3\sqrt{3}$, and each angle with an axis is $\cos^{-1}(1/\sqrt{3}) \approx 54.74^\circ$.

Answer: $3\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$ (or $-3\mathbf{i} - 3\mathbf{j} - 3\mathbf{k}$ if the angles are measured from the negative axes)

Q12. The forces $\mathbf{F}_1 = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ and $\mathbf{F}_2 = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ act together on a particle and displace it from $A(1, 1, 1)$ to $B(4, 3, 2)$. Find the total work done.

Step 1: Resultant force $\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 = 3\mathbf{i} + \mathbf{j} + 3\mathbf{k}$

Step 2: Displacement $\mathbf{d} = AB = (4 - 1)\mathbf{i} + (3 - 1)\mathbf{j} + (2 - 1)\mathbf{k} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$

Step 3: $W = \mathbf{R} \cdot \mathbf{d} = (3)(3) + (1)(2) + (3)(1) = 9 + 2 + 3 = 14$

Check: Work by $\mathbf{F}_1 = 6 + 6 + 1 = 13$ and work by $\mathbf{F}_2 = 3 - 4 + 2 = 1$. The sum is 14.

Answer: $W = 14$ joules

Q13. Two forces of magnitudes 5 N and 3 N act at a point with an angle of 60° between them. Find the magnitude of their resultant and of their difference.

Step 1: Let the forces be $\mathbf{P}$ and $\mathbf{Q}$, with $|\mathbf{P}| = 5$, $|\mathbf{Q}| = 3$ and $\mathbf{P} \cdot \mathbf{Q} = (5)(3) \cos 60^\circ = 15/2$.

Step 2 (resultant): $|\mathbf{P} + \mathbf{Q}|^2 = 25 + 9 + 2(15/2) = 49 \Rightarrow |\mathbf{P} + \mathbf{Q}| = 7$

Step 3 (difference): $|\mathbf{P} - \mathbf{Q}|^2 = 25 + 9 - 2(15/2) = 19 \Rightarrow |\mathbf{P} - \mathbf{Q}| = \sqrt{19}$

Answer: Resultant = 7 N; difference = $\sqrt{19}$ N ≈ 4.36 N

Q14. Prove the Cauchy–Schwarz inequality $|\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}||\mathbf{b}|$, and verify it for $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = 4\mathbf{i} - \mathbf{j} + 2\mathbf{k}$.

Step 1: $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$, so $|\mathbf{a} \cdot \mathbf{b}| = |\mathbf{a}||\mathbf{b}| |\cos \theta|$.

Step 2: For every angle, $|\cos \theta| \leq 1$, so $|\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}||\mathbf{b}|$. Equality holds only when $\cos \theta = \pm 1$, i.e. when the vectors are parallel.

Step 3 (verification): $\mathbf{a} \cdot \mathbf{b} = 4 - 2 + 6 = 8$, $|\mathbf{a}| = \sqrt{14}$, $|\mathbf{b}| = \sqrt{21}$, so $|\mathbf{a}||\mathbf{b}| = \sqrt{294} \approx 17.15$.

Answer: Proved. For the given vectors, $8 \leq 17.15$, so the inequality holds.

Q15. Prove the triangle inequality $|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$.

Step 1: $|\mathbf{a} + \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2\mathbf{a} \cdot \mathbf{b}$

Step 2: Since $\mathbf{a} \cdot \mathbf{b} \leq |\mathbf{a}||\mathbf{b}|$ (Cauchy–Schwarz), $|\mathbf{a} + \mathbf{b}|^2 \leq |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2|\mathbf{a}||\mathbf{b}| = (|\mathbf{a}| + |\mathbf{b}|)^2$

Step 3: Taking the square root of both sides (both are non-negative): $|a + b| \leq |a| + |b|$

Answer: Hence proved. **Meaning:** in any triangle, one side is not longer than the sum of the other two.

Q16. Using vectors, prove Pythagoras' theorem for a right-angled triangle.

Step 1: Let the triangle ABC have its right angle at C. Let $CA = a$ and $CB = b$. Since the angle at C is 90° , $a \cdot b = 0$.

Step 2: $AB = CB - CA = b - a$

Step 3: $|AB|^2 = |b - a|^2 = |a|^2 + |b|^2 - 2a \cdot b = |a|^2 + |b|^2 - 0$

Answer: $AB^2 = AC^2 + BC^2$. Hence proved.

Q17. In a triangle ABC, D is the midpoint of BC. Using vectors, prove that $AB^2 + AC^2 = 2(AD^2 + BD^2)$. (Apollonius' theorem)

Step 1: Take A as the origin. Let $AB = b$ and $AC = c$. Then $AD = (b + c)/2$ and $BD = AD - AB = (c - b)/2$.

Step 2: $AD^2 = |b + c|^2 / 4$ and $BD^2 = |c - b|^2 / 4$

Step 3: $AD^2 + BD^2 = (|b + c|^2 + |b - c|^2) / 4 = 2(|b|^2 + |c|^2) / 4$ (by the parallelogram law) $= (|b|^2 + |c|^2) / 2$

Step 4: So $2(AD^2 + BD^2) = |b|^2 + |c|^2 = AB^2 + AC^2$.

Answer: $AB^2 + AC^2 = 2(AD^2 + BD^2)$. Hence proved.

Q18. Using vectors, prove that the median drawn to the base of an isosceles triangle is perpendicular to the base.

Step 1: Let ABC be isosceles with $AB = AC$, and D the midpoint of the base BC. Take A as the origin, $AB = b$ and $AC = c$, so $|b| = |c|$.

Step 2: $AD = (b + c)/2$ and $BC = c - b$

Step 3: $AD \cdot BC = (1/2)(b + c) \cdot (c - b) = (1/2)(|c|^2 - |b|^2)$

Step 4: Since $|b| = |c|$, $AD \cdot BC = 0$.

Answer: AD is perpendicular to BC. Hence proved.

Q19. If a, b, c are three mutually perpendicular vectors of equal magnitude, prove that $a + b + c$ makes equal angles with a, b and c. Find that angle.

Step 1: Let $|a| = |b| = |c| = m$. Since they are mutually perpendicular, $a \cdot b = b \cdot c = c \cdot a = 0$.

Step 2: Let $d = a + b + c$. Then $|d|^2 = |a|^2 + |b|^2 + |c|^2 + 2(a \cdot b + b \cdot c + c \cdot a) = 3m^2$, so $|d| = \sqrt{3}m$.

Step 3: $d \cdot a = a \cdot a + b \cdot a + c \cdot a = m^2$. So $\cos \alpha = (d \cdot a) / (|d||a|) = m^2 / (\sqrt{3}m \cdot m) = 1/\sqrt{3}$.

Step 4: In the same way, $d \cdot b = m^2$ and $d \cdot c = m^2$, so the cosines of the angles with b and c are also $1/\sqrt{3}$.

Answer: The angle with each of a, b, c is $\cos^{-1}(1/\sqrt{3}) \approx 54.74^\circ$. Hence proved.

Q20. If a, b, c are unit vectors such that $a + b + c = 0$, find the value of $a \cdot b + b \cdot c + c \cdot a$.

Step 1: Since $a + b + c = 0$, we have $(a + b + c) \cdot (a + b + c) = 0$.

Step 2: $|a|^2 + |b|^2 + |c|^2 + 2(a \cdot b + b \cdot c + c \cdot a) = 0$

Step 3: All three are unit vectors, so $1 + 1 + 1 + 2(a \cdot b + b \cdot c + c \cdot a) = 0$.

Step 4: $2(a \cdot b + b \cdot c + c \cdot a) = -3$

Answer: $a \cdot b + b \cdot c + c \cdot a = -3/2$

Poly Notes Hub