

MATHEMATICS 1

Chapter: Vector – Scalar and Dot Product

Long Answer Questions with Solutions (Set 3)

This is a third set of important long-answer questions from the chapter for engineering students. Each question has step-by-step working and the final answer.

Q1. Find the angle between the vectors $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ and $\mathbf{b} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$.

Step 1: $\mathbf{a} \cdot \mathbf{b} = (1)(3) + (2)(-1) + (-3)(2) = 3 - 2 - 6 = -5$

Step 2: $|\mathbf{a}| = \sqrt{(1)^2 + (2)^2 + (-3)^2} = \sqrt{14}$, $|\mathbf{b}| = \sqrt{(3)^2 + (-1)^2 + (2)^2} = \sqrt{14}$

Step 3: $\cos \theta = -5 / (\sqrt{14} \times \sqrt{14}) = -5/14$

Step 4: The dot product is negative, so the angle is obtuse.

Answer: $\theta = \cos^{-1}(-5/14) \approx 110.92^\circ$

Q2. Find the angle between the vectors $\mathbf{a} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = (\sqrt{3} - 1)\mathbf{i} - (\sqrt{3} + 1)\mathbf{j} + 4\mathbf{k}$.

Step 1: $\mathbf{a} \cdot \mathbf{b} = (1)(\sqrt{3} - 1) + (1)(-(\sqrt{3} + 1)) + (2)(4) = \sqrt{3} - 1 - \sqrt{3} - 1 + 8 = 6$

Step 2: $|\mathbf{a}| = \sqrt{(1)^2 + (1)^2 + (2)^2} = \sqrt{6}$

Step 3: $|\mathbf{b}|^2 = (\sqrt{3} - 1)^2 + (-\sqrt{3} - 1)^2 + 4^2 = (4 - 2\sqrt{3}) + (4 + 2\sqrt{3}) + 16 = 24$, so $|\mathbf{b}| = 2\sqrt{6}$

Step 4: $\cos \theta = 6 / (\sqrt{6} \times 2\sqrt{6}) = 6/12 = 1/2$

Answer: $\theta = 60^\circ$

Q3. Find the value of λ for which the vectors $\mathbf{a} = \lambda\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} - 4\mathbf{j} - 6\mathbf{k}$ are (i) perpendicular, (ii) parallel.

(i) Perpendicular: $\mathbf{a} \cdot \mathbf{b} = 0$

$$2\lambda - 8 - 18 = 0 \Rightarrow 2\lambda = 26 \Rightarrow \lambda = 13$$

(ii) Parallel: $\mathbf{a} = m\mathbf{b}$ for some scalar m . Comparing the $\mathbf{j}$ components: $2 = -4m$, so $m = -1/2$.

The $\mathbf{k}$ components agree: $3 = -6m = 3$. Comparing the $\mathbf{i}$ components: $\lambda = 2m = -1$.

Answer: (i) $\lambda = 13$; (ii) $\lambda = -1$

Q4. Show that for any vector $\mathbf{a}$, $\mathbf{a} = (a \cdot \mathbf{i})\mathbf{i} + (a \cdot \mathbf{j})\mathbf{j} + (a \cdot \mathbf{k})\mathbf{k}$, and hence $|\mathbf{a}|^2 = (a \cdot \mathbf{i})^2 + (a \cdot \mathbf{j})^2 + (a \cdot \mathbf{k})^2$. Verify for $\mathbf{a} = 3\mathbf{i} - 4\mathbf{j} + 12\mathbf{k}$.

Step 1: Let $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$. Then $\mathbf{a} \cdot \mathbf{i} = a_1(\mathbf{i} \cdot \mathbf{i}) + a_2(\mathbf{j} \cdot \mathbf{i}) + a_3(\mathbf{k} \cdot \mathbf{i}) = a_1$.

Step 2: In the same way, $\mathbf{a} \cdot \mathbf{j} = a_2$ and $\mathbf{a} \cdot \mathbf{k} = a_3$. So $\mathbf{a} = (a \cdot \mathbf{i})\mathbf{i} + (a \cdot \mathbf{j})\mathbf{j} + (a \cdot \mathbf{k})\mathbf{k}$.

Step 3: $|\mathbf{a}|^2 = a_1^2 + a_2^2 + a_3^2 = (a \cdot \mathbf{i})^2 + (a \cdot \mathbf{j})^2 + (a \cdot \mathbf{k})^2$.

Step 4 (verification): For $\mathbf{a} = 3\mathbf{i} - 4\mathbf{j} + 12\mathbf{k}$: $\mathbf{a} \cdot \mathbf{i} = 3$, $\mathbf{a} \cdot \mathbf{j} = -4$, $\mathbf{a} \cdot \mathbf{k} = 12$. The sum of squares = $9 + 16 + 144 = 169$, so $|\mathbf{a}| = 13$.

Answer: Hence proved. For the given vector, $|\mathbf{a}| = 13$.

Q5. Show that the points A(2, 3, 5), B(1, 2, 3) and C(2, 3, 2) are the vertices of a right-angled triangle, and find its area.

Step 1: $\mathbf{BA} = (2 - 1)\mathbf{i} + (3 - 2)\mathbf{j} + (5 - 3)\mathbf{k} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$, $\mathbf{BC} = (2 - 1)\mathbf{i} + (3 - 2)\mathbf{j} + (2 - 3)\mathbf{k} = \mathbf{i} + \mathbf{j} - \mathbf{k}$

Step 2: $\mathbf{BA} \cdot \mathbf{BC} = 1 + 1 - 2 = 0$, so the angle at B is 90° .

Step 3: $|BA| = \sqrt{6}$ and $|BC| = \sqrt{3}$. Area = $(1/2)|BA||BC| = (1/2)\sqrt{18} = 3\sqrt{2}/2$.

Check: $AC = -3k$, so $AC^2 = 9 = 6 + 3 = BA^2 + BC^2$.

Answer: Right angle at B; area = $3\sqrt{2}/2 \approx 2.12$ square units

Q6. Show that the points A(1, 1, 1), B(2, 3, 3), C(4, 4, 1) and D(3, 2, -1) are the vertices of a square.

Step 1: $AB = i + 2j + 2k$ and $DC = (4 - 3)i + (4 - 2)j + (1 + 1)k = i + 2j + 2k$. Since $AB = DC$, ABCD is a parallelogram.

Step 2: $AD = 2i + j - 2k$. $AB \cdot AD = 2 + 2 - 4 = 0$, so adjacent sides are perpendicular. The parallelogram is a rectangle.

Step 3: $|AB| = \sqrt{(1 + 4 + 4)} = 3$ and $|AD| = \sqrt{(4 + 1 + 4)} = 3$. Adjacent sides are equal, so the rectangle is a square.

Check: $AC = 3i + 3j$ and $BD = i - j - 4k$. $AC \cdot BD = 3 - 3 = 0$ (perpendicular diagonals) and $|AC| = |BD| = \sqrt{18}$ (equal diagonals).

Answer: ABCD is a square. Hence proved.

Q7. Find the foot of the perpendicular and the perpendicular distance from the point P(1, 2, 3) to the line through the origin in the direction of $a = i + j + k$.

Step 1: The position vector of P is $p = i + 2j + 3k$. $p \cdot a = 1 + 2 + 3 = 6$ and $|a|^2 = 3$.

Step 2: The foot F is the vector projection of p on a : $OF = [(p \cdot a) / |a|^2] a = (6/3)(i + j + k) = 2i + 2j + 2k$.

Step 3: $PF = OF - p = i + 0j - k$. Check: $PF \cdot a = 1 + 0 - 1 = 0$, so PF is perpendicular to the line.

Step 4: Distance = $|PF| = \sqrt{(1 + 0 + 1)} = \sqrt{2}$.

Answer: Foot of the perpendicular F(2, 2, 2); distance = $\sqrt{2}$

Q8. For $a = i + 2j + 2k$ and $b = i + j + k$, find the value of λ for which $|a + \lambda b|$ is minimum, and find the minimum value.

Step 1: $|a + \lambda b|^2 = |a|^2 + 2\lambda(a \cdot b) + \lambda^2|b|^2 = 9 + 10\lambda + 3\lambda^2$

Step 2: This quadratic is smallest when its derivative is zero: $10 + 6\lambda = 0$, so $\lambda = -5/3$.

Step 3: Minimum value of the square = $9 - 50/3 + 25/3 = 9 - 25/3 = 2/3$. So the minimum of $|a + \lambda b|$ is $\sqrt{(2/3)} = \sqrt{6}/3$.

Check: For $\lambda = -5/3$, $(a + \lambda b) \cdot b = a \cdot b + \lambda|b|^2 = 5 - 5 = 0$. The shortest vector $a + \lambda b$ is perpendicular to b .

Answer: $\lambda = -5/3$; minimum value = $\sqrt{6}/3 \approx 0.816$

Q9. If $|a| = 3$, $|b| = 4$ and the angle between a and b is 60° , show that $2a - b$ is perpendicular to $a + 3b$.

Step 1: $a \cdot b = (3)(4) \cos 60^\circ = 6$

Step 2: $(2a - b) \cdot (a + 3b) = 2a \cdot a + 6a \cdot b - b \cdot a - 3b \cdot b = 2|a|^2 + 5a \cdot b - 3|b|^2$

Step 3: $= 2(9) + 5(6) - 3(16) = 18 + 30 - 48 = 0$

Answer: The dot product is 0, so $2a - b$ is perpendicular to $a + 3b$. Hence proved.

Q10. If $\mathbf{a}$ and $\mathbf{b}$ are unit vectors and $|\mathbf{a} + \mathbf{b}| = \sqrt{3}$, find the value of $(2\mathbf{a} - 5\mathbf{b}) \cdot (3\mathbf{a} + \mathbf{b})$.

Step 1: $|\mathbf{a} + \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2\mathbf{a} \cdot \mathbf{b} \Rightarrow 3 = 1 + 1 + 2\mathbf{a} \cdot \mathbf{b} \Rightarrow \mathbf{a} \cdot \mathbf{b} = 1/2$

Step 2: $(2\mathbf{a} - 5\mathbf{b}) \cdot (3\mathbf{a} + \mathbf{b}) = 6\mathbf{a} \cdot \mathbf{a} + 2\mathbf{a} \cdot \mathbf{b} - 15\mathbf{b} \cdot \mathbf{a} - 5\mathbf{b} \cdot \mathbf{b} = 6|\mathbf{a}|^2 - 13\mathbf{a} \cdot \mathbf{b} - 5|\mathbf{b}|^2$

Step 3: $= 6 - 13(1/2) - 5 = 6 - 6.5 - 5 = -11/2$

Answer: $-11/2$

Q11. If $|\mathbf{a}| = 3$ and $|\mathbf{b}| = 5$, find the values of λ for which $\mathbf{a} + \lambda\mathbf{b}$ is perpendicular to $\mathbf{a} - \lambda\mathbf{b}$.

Step 1: Perpendicular vectors satisfy $(\mathbf{a} + \lambda\mathbf{b}) \cdot (\mathbf{a} - \lambda\mathbf{b}) = 0$.

Step 2: $|\mathbf{a}|^2 - \lambda^2|\mathbf{b}|^2 = 0 \Rightarrow 9 - 25\lambda^2 = 0$

Step 3: $\lambda^2 = 9/25 \Rightarrow \lambda = \pm 3/5$

Answer: $\lambda = 3/5$ or $\lambda = -3/5$

Q12. If $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$ and $\mathbf{a} \neq \mathbf{0}$, does it follow that $\mathbf{b} = \mathbf{c}$? Explain with an example.

Step 1: $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$ gives $\mathbf{a} \cdot (\mathbf{b} - \mathbf{c}) = 0$.

Step 2: This only tells us that $\mathbf{b} - \mathbf{c}$ is perpendicular to $\mathbf{a}$ (or $\mathbf{b} - \mathbf{c} = \mathbf{0}$). It does not force $\mathbf{b} = \mathbf{c}$.

Step 3 (example): Take $\mathbf{a} = \mathbf{i}$, $\mathbf{b} = \mathbf{i} + \mathbf{j}$ and $\mathbf{c} = \mathbf{i} + \mathbf{k}$. Then $\mathbf{a} \cdot \mathbf{b} = 1$ and $\mathbf{a} \cdot \mathbf{c} = 1$, but $\mathbf{b} \neq \mathbf{c}$. Here $\mathbf{b} - \mathbf{c} = \mathbf{j} - \mathbf{k}$, which is perpendicular to $\mathbf{a}$.

Answer: No. The cancellation law does not hold for the dot product.

Q13. Prove that the vector $\mathbf{c} = |\mathbf{a}|\mathbf{b} + |\mathbf{b}|\mathbf{a}$ bisects the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$.

Step 1: $\mathbf{c} \cdot \mathbf{a} = |\mathbf{a}|(\mathbf{a} \cdot \mathbf{b}) + |\mathbf{b}||\mathbf{a}|^2 = |\mathbf{a}|(\mathbf{a} \cdot \mathbf{b} + |\mathbf{a}||\mathbf{b}|)$

Step 2: $\mathbf{c} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|^2 + |\mathbf{b}|(\mathbf{a} \cdot \mathbf{b}) = |\mathbf{b}|(\mathbf{a} \cdot \mathbf{b} + |\mathbf{a}||\mathbf{b}|)$

Step 3: Let α and β be the angles of $\mathbf{c}$ with $\mathbf{a}$ and $\mathbf{b}$. Then $\cos \alpha = (\mathbf{c} \cdot \mathbf{a}) / (|\mathbf{c}||\mathbf{a}|) = (\mathbf{a} \cdot \mathbf{b} + |\mathbf{a}||\mathbf{b}|) / |\mathbf{c}|$.

Step 4: In the same way, $\cos \beta = (\mathbf{c} \cdot \mathbf{b}) / (|\mathbf{c}||\mathbf{b}|) = (\mathbf{a} \cdot \mathbf{b} + |\mathbf{a}||\mathbf{b}|) / |\mathbf{c}|$.

Answer: $\cos \alpha = \cos \beta$, so $\alpha = \beta$. Hence $\mathbf{c}$ bisects the angle between $\mathbf{a}$ and $\mathbf{b}$.

Q14. The vectors $\mathbf{a}$ and $\mathbf{b}$ are such that $\mathbf{a} + 3\mathbf{b}$ is perpendicular to $7\mathbf{a} - 5\mathbf{b}$, and $\mathbf{a} - 4\mathbf{b}$ is perpendicular to $7\mathbf{a} - 2\mathbf{b}$. Find the angle between $\mathbf{a}$ and $\mathbf{b}$.

Step 1: $(\mathbf{a} + 3\mathbf{b}) \cdot (7\mathbf{a} - 5\mathbf{b}) = 0 \Rightarrow 7|\mathbf{a}|^2 + 16\mathbf{a} \cdot \mathbf{b} - 15|\mathbf{b}|^2 = 0 \dots(1)$

Step 2: $(\mathbf{a} - 4\mathbf{b}) \cdot (7\mathbf{a} - 2\mathbf{b}) = 0 \Rightarrow 7|\mathbf{a}|^2 - 30\mathbf{a} \cdot \mathbf{b} + 8|\mathbf{b}|^2 = 0 \dots(2)$

Step 3: Subtracting (2) from (1): $46\mathbf{a} \cdot \mathbf{b} - 23|\mathbf{b}|^2 = 0$, so $\mathbf{a} \cdot \mathbf{b} = |\mathbf{b}|^2/2$.

Step 4: Putting this in (1): $7|\mathbf{a}|^2 + 8|\mathbf{b}|^2 - 15|\mathbf{b}|^2 = 0$, so $|\mathbf{a}| = |\mathbf{b}|$.

Step 5: $\cos \theta = (\mathbf{a} \cdot \mathbf{b}) / (|\mathbf{a}||\mathbf{b}|) = (|\mathbf{b}|^2/2) / |\mathbf{b}|^2 = 1/2$

Answer: $\theta = 60^\circ$

Q15. A constant force $\mathbf{F} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ moves a particle from $P(1, 2, 3)$ to $Q(3, -1, 4)$ and then from Q to $R(5, 0, 5)$. Find the work done on each part and in total. Compare with the work done in going directly from P to R .

Step 1: $\mathbf{PQ} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$, $\mathbf{QR} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{PR} = 4\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$

Step 2: $W(P \text{ to } Q) = \mathbf{F} \cdot \mathbf{PQ} = 4 + 9 + 1 = 14$

Step 3: $W(Q \text{ to } R) = \mathbf{F} \cdot \mathbf{QR} = 4 - 3 + 1 = 2$. Total = $14 + 2 = 16$.

Step 4: Direct path: $F \cdot PR = 8 + 6 + 2 = 16$.

Answer: Total work = 16 joules, the same as the direct path. For a constant force, work depends only on the end points because $PQ + QR = PR$.

Q16. A force of 20 N acts at an angle of 30° to the horizontal and moves a body 5 m along the horizontal. Find the work done by the force.

Step 1: Work done $W = F \cdot d = |F||d| \cos \theta$

Step 2: $W = (20)(5) \cos 30^\circ = 100 \times (\sqrt{3}/2) = 50\sqrt{3}$

Answer: $W = 50\sqrt{3} \approx 86.6$ joules

Q17. A force $F = 3i + 4j - 2k$ newtons acts on a body moving with velocity $v = 2i - j + 5k$ m/s. Find the power delivered by the force. (Power = $F \cdot v$)

Step 1: $P = F \cdot v = (3)(2) + (4)(-1) + (-2)(5)$

Step 2: $P = 6 - 4 - 10 = -8$

Step 3: The negative sign means that, overall, the force acts against the motion and takes energy from the body.

Answer: $P = -8$ watts

Q18. A block of weight 50 N rests on a smooth plane inclined at 30° to the horizontal. Using the dot product, find the components of the weight parallel and perpendicular to the plane.

Step 1: Take x horizontal and y vertically up. Weight $W = -50j$.

Step 2: Unit vector up the slope: $u = \cos 30^\circ i + \sin 30^\circ j = (\sqrt{3}/2)i + (1/2)j$. Unit vector normal to the plane: $n = -(1/2)i + (\sqrt{3}/2)j$ (check: $u \cdot n = 0$).

Step 3: Parallel component = $W \cdot u = -50 \times (1/2) = -25$. Its magnitude is 25 N, and the negative sign shows it acts down the slope.

Step 4: Perpendicular component = $W \cdot n = -50 \times (\sqrt{3}/2) = -25\sqrt{3}$. Its magnitude is about 43.3 N, acting into the plane.

Check: $25^2 + (25\sqrt{3})^2 = 625 + 1875 = 2500 = 50^2$.

Answer: Parallel to the plane: 25 N (down the slope); perpendicular to the plane: $25\sqrt{3} \approx 43.3$ N

Q19. Using vectors, prove that the three altitudes of a triangle meet at a point.

Step 1: Let the altitudes from A and B meet at H. Take H as the origin, with $HA = a$, $HB = b$ and $HC = c$.

Step 2: AH is perpendicular to BC, so $a \cdot (c - b) = 0$, which gives $a \cdot c = a \cdot b \dots(1)$

Step 3: BH is perpendicular to CA, so $b \cdot (a - c) = 0$, which gives $b \cdot a = b \cdot c \dots(2)$

Step 4: From (1) and (2), $a \cdot c = b \cdot c$, so $c \cdot (a - b) = 0$. Since $AB = b - a$, this says $HC \cdot AB = 0$.

Answer: HC is perpendicular to AB, so H also lies on the altitude from C. The three altitudes meet at H (the orthocentre). Hence proved.

Q20. In a quadrilateral ABCD the diagonals AC and BD are perpendicular. Prove that $AB^2 + CD^2 = BC^2 + DA^2$.

Step 1: Let $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ be the position vectors of A, B, C, D. Then $AB^2 + CD^2 - BC^2 - DA^2 = |\mathbf{b} - \mathbf{a}|^2 + |\mathbf{d} - \mathbf{c}|^2 - |\mathbf{c} - \mathbf{b}|^2 - |\mathbf{a} - \mathbf{d}|^2$.

Step 2: Expand each term: $(|\mathbf{b}|^2 - 2\mathbf{a} \cdot \mathbf{b} + |\mathbf{a}|^2) + (|\mathbf{d}|^2 - 2\mathbf{c} \cdot \mathbf{d} + |\mathbf{c}|^2) - (|\mathbf{c}|^2 - 2\mathbf{b} \cdot \mathbf{c} + |\mathbf{b}|^2) - (|\mathbf{a}|^2 - 2\mathbf{a} \cdot \mathbf{d} + |\mathbf{d}|^2)$

Step 3: All the squared terms cancel, leaving $-2\mathbf{a} \cdot \mathbf{b} - 2\mathbf{c} \cdot \mathbf{d} + 2\mathbf{b} \cdot \mathbf{c} + 2\mathbf{a} \cdot \mathbf{d} = 2\mathbf{a} \cdot (\mathbf{d} - \mathbf{b}) - 2\mathbf{c} \cdot (\mathbf{d} - \mathbf{b}) = 2(\mathbf{a} - \mathbf{c}) \cdot (\mathbf{d} - \mathbf{b})$.

Step 4: Now $\mathbf{a} - \mathbf{c} = -\mathbf{AC}$ and $\mathbf{d} - \mathbf{b} = \mathbf{BD}$, so $AB^2 + CD^2 - BC^2 - DA^2 = -2 \mathbf{AC} \cdot \mathbf{BD}$.

Step 5: The diagonals are perpendicular, so $\mathbf{AC} \cdot \mathbf{BD} = 0$.

Answer: $AB^2 + CD^2 = BC^2 + DA^2$. Hence proved.