

MATHEMATICS 1

Chapter: Vector – Scalar and Dot Product

Long Answer Questions with Solutions (Set 4)

This is a fourth set of important long-answer questions from the chapter for engineering students. Each question has step-by-step working and the final answer.

Q1. If $\mathbf{a} = 5\mathbf{i} - \mathbf{j} - 3\mathbf{k}$ and $\mathbf{b} = \mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$, show that the vectors $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$ are perpendicular.

Step 1: $\mathbf{a} + \mathbf{b} = 6\mathbf{i} + 2\mathbf{j} - 8\mathbf{k}$ and $\mathbf{a} - \mathbf{b} = 4\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$

Step 2: $(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = (6)(4) + (2)(-4) + (-8)(2) = 24 - 8 - 16 = 0$

Step 3 (alternative): In general $(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = |\mathbf{a}|^2 - |\mathbf{b}|^2$. Here $|\mathbf{a}|^2 = 25 + 1 + 9 = 35$ and $|\mathbf{b}|^2 = 1 + 9 + 25 = 35$, and these are equal.

Answer: The dot product is 0, so $\mathbf{a} + \mathbf{b}$ is perpendicular to $\mathbf{a} - \mathbf{b}$. Hence proved.

Q2. Find the value of m for which the vectors $\mathbf{a} = 2\mathbf{i} - m\mathbf{j} + \mathbf{k}$ and $\mathbf{b} = \mathbf{i} + 2\mathbf{j} - m\mathbf{k}$ make an angle of 90° .

Step 1: For a 90° angle, $\mathbf{a} \cdot \mathbf{b} = 0$.

Step 2: $(2)(1) + (-m)(2) + (1)(-m) = 0 \Rightarrow 2 - 2m - m = 0 \Rightarrow 2 - 3m = 0$

Step 3: $m = 2/3$

Answer: $m = 2/3$

Q3. Find the values of x for which the angle between the vectors $\mathbf{a} = 2x^2\mathbf{i} + 4x\mathbf{j} + \mathbf{k}$ and $\mathbf{b} = 7\mathbf{i} - 2\mathbf{j} + x\mathbf{k}$ is acute, and the values for which it is a right angle.

Step 1: $\mathbf{a} \cdot \mathbf{b} = 14x^2 - 8x + x = 14x^2 - 7x = 7x(2x - 1)$

Step 2 (right angle): $\mathbf{a} \cdot \mathbf{b} = 0$ gives $x = 0$ or $x = 1/2$.

Step 3 (acute): For an acute angle, $\mathbf{a} \cdot \mathbf{b} > 0$, so $7x(2x - 1) > 0$, which holds when $x < 0$ or $x > 1/2$.

Step 4: As in a similar earlier problem, checking whether the vectors could point in exactly opposite directions shows this does not happen for real x , so the inequality alone decides the acute case.

Answer: Right angle: $x = 0$ or $x = 1/2$; acute angle: $x < 0$ or $x > 1/2$

Q4. If A, B, C are the vertices of a triangle with position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$, show that $(\mathbf{b} - \mathbf{a}) \cdot (\mathbf{c} - \mathbf{a}) = |\mathbf{AB}||\mathbf{AC}| \cos A$, and use this to find angle A when $A(1, 0, 0), B(0, 1, 0), C(0, 0, 1)$.

Step 1: $\mathbf{AB} = \mathbf{b} - \mathbf{a} = -\mathbf{i} + \mathbf{j}$ and $\mathbf{AC} = \mathbf{c} - \mathbf{a} = -\mathbf{i} + \mathbf{k}$

Step 2: $\mathbf{AB} \cdot \mathbf{AC} = (-1)(-1) + (1)(0) + (0)(1) = 1$

Step 3: $|\mathbf{AB}| = \sqrt{2}$ and $|\mathbf{AC}| = \sqrt{2}$

Step 4: $\cos A = (\mathbf{AB} \cdot \mathbf{AC}) / (|\mathbf{AB}||\mathbf{AC}|) = 1/2$

Answer: $A = 60^\circ$

Q5. Show that the points A(1, -2, 3), B(2, 3, -4) and C(0, -7, 10) are collinear, using the dot product.

Step 1: $AB = i + 5j - 7k$ and $AC = -i - 5j + 7k$

Step 2: Notice $AC = -AB$, i.e. AC is a scalar multiple of AB (with scalar -1).

Step 3: Since AB and AC are parallel and share the point A , the points A, B, C lie on the same straight line. (A dot-product check: $AB \cdot AC = -1 - 25 - 49 = -75$, and $|AB| = |AC| = \sqrt{75}$, so $\cos \theta = -75/75 = -1$, giving $\theta = 180^\circ$, confirming A, B, C are collinear with B and C on opposite sides of A .)

Answer: A, B, C are collinear (B and C lie on opposite sides of A).

Q6. If $a + b + c = 0$, $|a| = 3$, $|b| = 4$, $|c| = \sqrt{37}$, find the angle between b and c .

Step 1: $b + c = -a$, so $|b + c|^2 = |a|^2$

Step 2: $|b|^2 + |c|^2 + 2b \cdot c = |a|^2 \Rightarrow 16 + 37 + 2b \cdot c = 9 \Rightarrow b \cdot c = -22$

Step 3: $\cos \theta = (b \cdot c) / (|b||c|) = -22 / (4\sqrt{37})$

Answer: $\theta = \cos^{-1}(-22 / (4\sqrt{37})) \approx 154.7^\circ$

Q7. If $|a| = 4$, $|b| = 3$ and the angle between a and b is $2\pi/3$, find $|a + b|$ and the angle it makes with a .

Step 1: $a \cdot b = (4)(3) \cos 120^\circ = 12 \times (-1/2) = -6$

Step 2: $|a + b|^2 = 16 + 9 + 2(-6) = 13$, so $|a + b| = \sqrt{13}$

Step 3: $(a + b) \cdot a = |a|^2 + a \cdot b = 16 - 6 = 10$

Step 4: $\cos \phi = 10 / (\sqrt{13} \times 4) = 10 / (4\sqrt{13}) = 5 / (2\sqrt{13})$

Answer: $|a + b| = \sqrt{13}$; angle with $a = \cos^{-1}(5/(2\sqrt{13})) \approx 46.1^\circ$

Q8. If a, b, c are mutually perpendicular vectors with $|a| = 1$, $|b| = 2$, $|c| = 3$, find $|a + b + c|$ and the angle it makes with c .

Step 1: Since a, b, c are mutually perpendicular, $a \cdot b = b \cdot c = c \cdot a = 0$.

Step 2: $|a + b + c|^2 = |a|^2 + |b|^2 + |c|^2 = 1 + 4 + 9 = 14$, so $|a + b + c| = \sqrt{14}$

Step 3: $(a + b + c) \cdot c = |c|^2 = 9$ (the other dot products vanish)

Step 4: $\cos \phi = 9 / (\sqrt{14} \times 3) = 3/\sqrt{14}$

Answer: $|a + b + c| = \sqrt{14}$; angle with $c = \cos^{-1}(3/\sqrt{14}) \approx 36.7^\circ$

Q9. Find the value of $(a + b + c) \cdot (a + b + c)$ if $|a| = |b| = |c| = 1$ and $a \cdot b = b \cdot c = c \cdot a = -1/2$.

Step 1: $(a + b + c) \cdot (a + b + c) = |a|^2 + |b|^2 + |c|^2 + 2(a \cdot b + b \cdot c + c \cdot a)$

Step 2: $= 1 + 1 + 1 + 2(-1/2 - 1/2 - 1/2) = 3 + 2(-3/2) = 3 - 3$

Answer: 0 , i.e. $|a + b + c| = 0$, so $a + b + c = 0$

Q10. Find the value of p for which the points $A(3, 2, 1)$, $B(4, p, 5)$ and $C(4, 2, -2)$ form a right angle at A .

Step 1: $AB = i + (p - 2)j + 4k$ and $AC = i + 0j - 3k$

Step 2: For a right angle at A , $AB \cdot AC = 0$.

Step 3: $(1)(1) + (p - 2)(0) + (4)(-3) = 0 \Rightarrow 1 - 12 = -11 \neq 0$ for any p

Step 4: Since the j -terms drop out, the equation $1 - 12 = 0$ has no solution, meaning no value of p can make AB and AC perpendicular here.

Answer: No such p exists; the given points can never form a right angle at A for any value of p .

Q11. Two adjacent sides of a parallelogram are given by $\mathbf{a} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = \mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$. Find the length of the diagonal that is the sum of the vectors, and the angle it makes with $\mathbf{a}$.

Step 1: Diagonal $\mathbf{d} = \mathbf{a} + \mathbf{b} = 4\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$, so $|\mathbf{d}| = \sqrt{(16 + 4 + 4)} = \sqrt{24} = 2\sqrt{6}$

Step 2: $\mathbf{d} \cdot \mathbf{a} = |\mathbf{a}|^2 + \mathbf{a} \cdot \mathbf{b}$. $|\mathbf{a}|^2 = 9 + 1 + 4 = 14$ and $\mathbf{a} \cdot \mathbf{b} = 3 - 3 - 8 = -8$, so $\mathbf{d} \cdot \mathbf{a} = 14 - 8 = 6$

Step 3: $|\mathbf{a}| = \sqrt{14}$, so $\cos \phi = 6 / (2\sqrt{6} \times \sqrt{14}) = 6 / (2\sqrt{84}) = 3/\sqrt{84}$

Answer: Diagonal length = $2\sqrt{6}$; angle with $\mathbf{a} = \cos^{-1}(3/\sqrt{84}) \approx 70.9^\circ$

Q12. If $\mathbf{a}$ and $\mathbf{b}$ are two vectors such that $(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = 0$ and $|\mathbf{a} + \mathbf{b}| = 2|\mathbf{a} - \mathbf{b}|$, find the ratio $|\mathbf{a}| : |\mathbf{b}|$.

Step 1: $(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = 0$ gives $|\mathbf{a}|^2 = |\mathbf{b}|^2 \dots (1)$

Step 2: $|\mathbf{a} + \mathbf{b}|^2 = 4|\mathbf{a} - \mathbf{b}|^2$ gives $|\mathbf{a}|^2 + |\mathbf{b}|^2 + 2\mathbf{a} \cdot \mathbf{b} = 4(|\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\mathbf{a} \cdot \mathbf{b})$

Step 3: Using (1), let $|\mathbf{a}|^2 = |\mathbf{b}|^2 = k$. Then $k + k + 2\mathbf{a} \cdot \mathbf{b} = 4(k + k - 2\mathbf{a} \cdot \mathbf{b}) \Rightarrow 2k + 2\mathbf{a} \cdot \mathbf{b} = 8k - 8\mathbf{a} \cdot \mathbf{b} \Rightarrow 10\mathbf{a} \cdot \mathbf{b} = 6k \Rightarrow \mathbf{a} \cdot \mathbf{b} = 3k/5$

Step 4: Since $\mathbf{a} \cdot \mathbf{b} \leq |\mathbf{a}||\mathbf{b}| = k$, we need $3k/5 \leq k$, which is true, so a consistent solution exists with $|\mathbf{a}| = |\mathbf{b}|$.

Answer: $|\mathbf{a}| : |\mathbf{b}| = 1 : 1$

Q13. Two forces $\mathbf{P}$ and $\mathbf{Q}$ act at a point so that their resultant $\mathbf{R} = \mathbf{P} + \mathbf{Q}$ is perpendicular to $\mathbf{P}$. If θ is the angle between $\mathbf{P}$ and $\mathbf{Q}$, prove that $|\mathbf{Q}| \cos \theta = -|\mathbf{P}|$.

Step 1: $\mathbf{R}$ is perpendicular to $\mathbf{P}$, so $\mathbf{R} \cdot \mathbf{P} = 0$, i.e. $(\mathbf{P} + \mathbf{Q}) \cdot \mathbf{P} = 0$.

Step 2: Expanding: $|\mathbf{P}|^2 + \mathbf{Q} \cdot \mathbf{P} = 0$, so $\mathbf{Q} \cdot \mathbf{P} = -|\mathbf{P}|^2$.

Step 3: Also $\mathbf{Q} \cdot \mathbf{P} = |\mathbf{Q}||\mathbf{P}| \cos \theta$, so $|\mathbf{Q}||\mathbf{P}| \cos \theta = -|\mathbf{P}|^2$.

Step 4: Dividing both sides by $|\mathbf{P}|$ (which is non-zero): $|\mathbf{Q}| \cos \theta = -|\mathbf{P}|$.

Answer: $|\mathbf{Q}| \cos \theta = -|\mathbf{P}|$. Hence proved.

Q14. A particle acted on by constant forces $\mathbf{F}_1 = 4\mathbf{i} + \mathbf{j} - 3\mathbf{k}$ and $\mathbf{F}_2 = 3\mathbf{i} + \mathbf{j} - \mathbf{k}$ is displaced from the point $(1, 2, 3)$ to the point $(5, 4, 1)$. Find the total work done.

Step 1: Resultant force $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 = 7\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$

Step 2: Displacement $\mathbf{d} = (5 - 1)\mathbf{i} + (4 - 2)\mathbf{j} + (1 - 3)\mathbf{k} = 4\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$

Step 3: $W = \mathbf{F} \cdot \mathbf{d} = (7)(4) + (2)(2) + (-4)(-2) = 28 + 4 + 8 = 40$

Answer: $W = 40$ joules

Q15. A ship is pulled by two ropes making angles of 30° and 45° with the direction of motion, with tensions of 200 N and 150 N. Find the total useful work done (component of each force along the direction of motion) when the ship moves 100 m.

Step 1: Useful component of the first rope = $200 \cos 30^\circ = 200 \times (\sqrt{3}/2) = 100\sqrt{3}$ N

Step 2: Useful component of the second rope = $150 \cos 45^\circ = 150 \times (\sqrt{2}/2) = 75\sqrt{2}$ N

Step 3: Total useful force = $100\sqrt{3} + 75\sqrt{2} \approx 173.2 + 106.1 = 279.3$ N

Step 4: Work done = Total force $\times$ distance = $279.3 \times 100 = 27930$ (approx.)

Answer: Work done $\approx 27,930$ joules

Q16. Using vectors, prove that in a parallelogram the sum of the squares of the diagonals is equal to twice the sum of the squares of the sides. Verify this for a parallelogram with adjacent sides $\mathbf{a} = 2\mathbf{i} + \mathbf{j}$ and $\mathbf{b} = \mathbf{i} + 2\mathbf{j}$.

Step 1: For adjacent sides $\mathbf{a}$, $\mathbf{b}$, the diagonals are $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$. $|\mathbf{a} + \mathbf{b}|^2 + |\mathbf{a} - \mathbf{b}|^2 = 2|\mathbf{a}|^2 + 2|\mathbf{b}|^2$ (shown earlier by expanding both squares and adding).

Step 2: The sum of squares of all four sides is $2|\mathbf{a}|^2 + 2|\mathbf{b}|^2$, so the identity says: sum of squares of diagonals = sum of squares of the four sides.

Step 3 (verification): $|\mathbf{a}|^2 = 5$, $|\mathbf{b}|^2 = 5$. $\mathbf{a} + \mathbf{b} = 3\mathbf{i} + 3\mathbf{j}$, so $|\mathbf{a} + \mathbf{b}|^2 = 18$. $\mathbf{a} - \mathbf{b} = \mathbf{i} - \mathbf{j}$, so $|\mathbf{a} - \mathbf{b}|^2 = 2$.

Step 4: Sum of squares of diagonals = $18 + 2 = 20$. Sum of squares of sides = $2(5) + 2(5) = 20$.

Answer: Both sides equal 20, confirming the identity. Hence proved and verified.

Q17. If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are the position vectors of A, B, C and $2\mathbf{a} + 3\mathbf{b} - 5\mathbf{c} = \mathbf{0}$, show that A, B, C are collinear and find the ratio in which C divides AB.

Step 1: From $2\mathbf{a} + 3\mathbf{b} - 5\mathbf{c} = \mathbf{0}$, we get $\mathbf{c} = (2\mathbf{a} + 3\mathbf{b})/5$.

Step 2: This is the section formula for a point dividing AB in the ratio 3:2 from A (weights 2 and 3 sum to 5), so C lies on the line AB, dividing it internally.

Step 3 (collinearity check): $\mathbf{AC} = \mathbf{c} - \mathbf{a} = (2\mathbf{a} + 3\mathbf{b} - 5\mathbf{a})/5 = (3\mathbf{b} - 3\mathbf{a})/5 = (3/5)(\mathbf{b} - \mathbf{a}) = (3/5)\mathbf{AB}$.

Step 4: Since AC is a scalar multiple of AB, the vectors are parallel and share the point A.

Answer: A, B, C are collinear, with C dividing AB internally in the ratio 3:2 from A.

Q18. If the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ satisfy $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$, $\mathbf{b} \cdot \mathbf{c} = \mathbf{b} \cdot \mathbf{a}$, and $\mathbf{c} \cdot \mathbf{a} = \mathbf{c} \cdot \mathbf{b}$, and none of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ is zero, show that $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{c} = \mathbf{c} \cdot \mathbf{a}$.

Step 1: From the first condition, $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c} \dots(1)$

Step 2: From the second condition, $\mathbf{b} \cdot \mathbf{c} = \mathbf{b} \cdot \mathbf{a} = \mathbf{a} \cdot \mathbf{b} \dots(2)$, using the commutativity of the dot product.

Step 3: From the third condition, $\mathbf{c} \cdot \mathbf{a} = \mathbf{c} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{c} \dots(3)$

Step 4: Combining (1), (2) and (3): $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c} = \mathbf{c} \cdot \mathbf{a}$, and $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{c}$. So all three products are equal.

Answer: $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{c} = \mathbf{c} \cdot \mathbf{a}$. Hence proved.

Q19. A vector $\mathbf{r}$ satisfies $\mathbf{r} \cdot \mathbf{i} = \mathbf{r} \cdot \mathbf{j} = \mathbf{r} \cdot \mathbf{k}$ and $|\mathbf{r}| = 3\sqrt{3}$. Find $\mathbf{r}$.

Step 1: Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Then $\mathbf{r} \cdot \mathbf{i} = x$, $\mathbf{r} \cdot \mathbf{j} = y$, $\mathbf{r} \cdot \mathbf{k} = z$. The condition $x = y = z$ means the components are equal.

Step 2: $|\mathbf{r}|^2 = x^2 + x^2 + x^2 = 3x^2 = 27$ (since $|\mathbf{r}|^2 = 27$)

Step 3: $x^2 = 9$, so $x = \pm 3$

Answer: $\mathbf{r} = 3\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$ or $\mathbf{r} = -3\mathbf{i} - 3\mathbf{j} - 3\mathbf{k}$