

MATHEMATICS 1

Chapter: Vector – Scalar and Dot Product

Exercise Notes: Important Questions with Solutions

These notes are prepared for engineering students. They cover the most important exam-oriented questions from the chapter, with step-by-step solutions. **Section A** has short-answer questions, **Section B** has long-answer questions, and **Section C** has practice questions with answers. A quick revision of the key formulas is given first.

Quick Revision: Key Points and Formulas

1. Scalar and Vector

A **scalar** has magnitude only (mass, time, speed, work, energy). A **vector** has both magnitude and direction (displacement, velocity, force, acceleration).

2. Vector in Component Form

- $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are unit vectors along the x, y and z axes.
- Magnitude: $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$
- Unit vector along $\mathbf{a} = \mathbf{a} / |\mathbf{a}|$
- Vector AB from A(x_1, y_1, z_1) to B(x_2, y_2, z_2) = $(x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}$ (final point – initial point)
- Direction cosines: $l = a_1/|\mathbf{a}|$, $m = a_2/|\mathbf{a}|$, $n = a_3/|\mathbf{a}|$, and $l^2 + m^2 + n^2 = 1$

3. Dot (Scalar) Product

The dot product of two vectors gives a **scalar**. If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ ($0^\circ \leq \theta \leq 180^\circ$):

- $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$
- In components: $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$
- Angle between two vectors: $\cos \theta = (\mathbf{a} \cdot \mathbf{b}) / (|\mathbf{a}||\mathbf{b}|)$
- If θ is acute, $\mathbf{a} \cdot \mathbf{b}$ is positive; if θ is obtuse, it is negative; if $\theta = 90^\circ$, it is zero.
- Two non-zero vectors are perpendicular if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.
- Two vectors are parallel when $\mathbf{a} = k\mathbf{b}$, and then $\mathbf{a} \cdot \mathbf{b} = \pm|\mathbf{a}||\mathbf{b}|$.

Dot products of the unit vectors:

.	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{i}$	1	0	0
$\mathbf{j}$	0	1	0
$\mathbf{k}$	0	0	1

4. Properties and Useful Results

- $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$ and $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
- $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$ and $(m\mathbf{a}) \cdot (n\mathbf{b}) = mn(\mathbf{a} \cdot \mathbf{b})$
- $|\mathbf{a} + \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2\mathbf{a} \cdot \mathbf{b}$

- $|\mathbf{a} - \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\mathbf{a} \cdot \mathbf{b}$
- $(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = |\mathbf{a}|^2 - |\mathbf{b}|^2$
- $|\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}||\mathbf{b}|$

5. Projection and Work Done

- Scalar projection of $\mathbf{a}$ on $\mathbf{b} = (\mathbf{a} \cdot \mathbf{b}) / |\mathbf{b}|$
- Vector projection of $\mathbf{a}$ on $\mathbf{b} = [(\mathbf{a} \cdot \mathbf{b}) / |\mathbf{b}|^2] \mathbf{b}$
- Work done by a constant force $\mathbf{F}$ for a displacement $\mathbf{d}$: $W = \mathbf{F} \cdot \mathbf{d}$

Section A: Short Answer Questions

A1. Define a scalar quantity and a vector quantity. Give two examples of each.

Solution: A **scalar** is a quantity that has magnitude only (no direction), e.g. mass and time.
A **vector** is a quantity that has both magnitude and direction, e.g. displacement and force.

A2. Find the magnitude of the vector $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$.

Solution: $|\mathbf{a}| = \sqrt{(2)^2 + (-3)^2 + 6^2} = \sqrt{4 + 9 + 36} = \sqrt{49}$

Answer: $|\mathbf{a}| = 7$

A3. Find the unit vector along $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j} - 12\mathbf{k}$.

Solution: $|\mathbf{a}| = \sqrt{(3)^2 + 4^2 + (-12)^2} = \sqrt{9 + 16 + 144} = \sqrt{169} = 13$

Unit vector = $\mathbf{a} / |\mathbf{a}| = (3\mathbf{i} + 4\mathbf{j} - 12\mathbf{k}) / 13$

Answer: $(3/13)\mathbf{i} + (4/13)\mathbf{j} - (12/13)\mathbf{k}$

A4. Find the vector $\mathbf{AB}$ and its magnitude, where $A(2, 3, 1)$ and $B(5, -1, 4)$.

Solution: $\mathbf{AB} = \text{position vector of } B - \text{position vector of } A$

$\mathbf{AB} = (5 - 2)\mathbf{i} + (-1 - 3)\mathbf{j} + (4 - 1)\mathbf{k} = 3\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}$

$|\mathbf{AB}| = \sqrt{(3)^2 + (-4)^2 + 3^2} = \sqrt{9 + 16 + 9} = \sqrt{34}$

Answer: $\mathbf{AB} = 3\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}$, $|\mathbf{AB}| = \sqrt{34} \approx 5.83$

A5. Write the values of $\mathbf{i} \cdot \mathbf{i}$, $\mathbf{j} \cdot \mathbf{j}$, $\mathbf{k} \cdot \mathbf{k}$, $\mathbf{i} \cdot \mathbf{j}$, $\mathbf{j} \cdot \mathbf{k}$ and $\mathbf{k} \cdot \mathbf{i}$.

Solution: Each of $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ has magnitude 1, and the angle between a unit vector and itself is 0° , so $\mathbf{i} \cdot \mathbf{i} = (1)(1)\cos 0^\circ = 1$. The three unit vectors are mutually perpendicular, so the dot product of two different ones is $(1)(1)\cos 90^\circ = 0$.

Answer: $\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1$; $\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0$

A6. Find $\mathbf{a} \cdot \mathbf{b}$ if $\mathbf{a} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}$.

Solution: $\mathbf{a} \cdot \mathbf{b} = (3)(2) + (-2)(4) + (1)(-5) = 6 - 8 - 5$

Answer: $\mathbf{a} \cdot \mathbf{b} = -7$

A7. Find the value of λ for which the vectors $\mathbf{a} = 2\mathbf{i} + \lambda\mathbf{j} + \mathbf{k}$ and $\mathbf{b} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ are perpendicular.

Solution: Perpendicular vectors satisfy $\mathbf{a} \cdot \mathbf{b} = 0$.

$(2)(1) + (\lambda)(-2) + (1)(3) = 0 \Rightarrow 5 - 2\lambda = 0$

Answer: $\lambda = 5/2$

A8. If $|\mathbf{a}| = 3$, $|\mathbf{b}| = 4$ and the angle between them is 60° , find $\mathbf{a} \cdot \mathbf{b}$.

Solution: $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta = (3)(4) \cos 60^\circ = 12 \times \frac{1}{2}$

Answer: $\mathbf{a} \cdot \mathbf{b} = 6$

A9. Find the angle between $\mathbf{a} = \mathbf{i} + \mathbf{j}$ and $\mathbf{b} = \mathbf{j} + \mathbf{k}$.

Solution: $\mathbf{a} \cdot \mathbf{b} = (1)(0) + (1)(1) + (0)(1) = 1$

$|\mathbf{a}| = \sqrt{2}$ and $|\mathbf{b}| = \sqrt{2}$, so $\cos \theta = 1 / (\sqrt{2} \times \sqrt{2}) = 1/2$

Answer: $\theta = 60^\circ$

A10. Find the projection of $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ on $\mathbf{b} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$.

Solution: $\mathbf{a} \cdot \mathbf{b} = 2 + 6 + 2 = 10$ and $|\mathbf{b}| = \sqrt{(1 + 4 + 4)} = 3$

Projection of $\mathbf{a}$ on $\mathbf{b} = (\mathbf{a} \cdot \mathbf{b}) / |\mathbf{b}| = 10/3$

Answer: $10/3$

A11. Define direction cosines of a vector and state the relation among them.

Solution: If a vector makes angles α , β , γ with the positive x, y and z axes, then $l = \cos \alpha$, $m = \cos \beta$, $n = \cos \gamma$ are called its direction cosines. For $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$:

- $l = a_1/|\mathbf{a}|$, $m = a_2/|\mathbf{a}|$, $n = a_3/|\mathbf{a}|$

Squaring and adding gives $(a_1^2 + a_2^2 + a_3^2)/|\mathbf{a}|^2 = 1$.

Answer: $l^2 + m^2 + n^2 = 1$

A12. If $\mathbf{a} \cdot \mathbf{b} = 0$, does it mean that $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$? Explain.

Solution: $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$. This is zero when $|\mathbf{a}| = 0$, or $|\mathbf{b}| = 0$, or $\cos \theta = 0$, i.e. $\theta = 90^\circ$. So two non-zero perpendicular vectors also have a zero dot product (for example $\mathbf{i} \cdot \mathbf{j} = 0$).

Answer: No. $\mathbf{a} \cdot \mathbf{b} = 0$ also when the non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ are perpendicular.

A13. State any four properties of the dot product.

Solution:

- Commutative: $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
- Distributive: $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
- $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$
- $(m\mathbf{a}) \cdot (n\mathbf{b}) = mn(\mathbf{a} \cdot \mathbf{b})$, where m and n are scalars
- Two non-zero vectors are perpendicular if and only if $\mathbf{a} \cdot \mathbf{b} = 0$

A14. Is the expression $(\mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{c}$ meaningful? Give a reason.

Solution: $\mathbf{a} \cdot \mathbf{b}$ is a scalar. A dot product is defined only between two vectors, so (scalar) $\cdot \mathbf{c}$ has no meaning. (By contrast, $(\mathbf{a} \cdot \mathbf{b})\mathbf{c}$ is meaningful; it is a vector along $\mathbf{c}$.)

Answer: No, it is not meaningful.

A15. Give the geometrical meaning of the dot product $\mathbf{a} \cdot \mathbf{b}$.

Solution: $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| (|\mathbf{b}| \cos \theta) = |\mathbf{a}| \times (\text{projection of } \mathbf{b} \text{ on } \mathbf{a})$.

Similarly, $\mathbf{a} \cdot \mathbf{b} = |\mathbf{b}| \times (\text{projection of } \mathbf{a} \text{ on } \mathbf{b})$. So the dot product is the length of one vector multiplied by the component of the other vector along it.

Section B: Long Answer Questions

B1. Find the angle between the vectors $\mathbf{a} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{b} = \mathbf{i} + \mathbf{j}$.

Step 1: $\mathbf{a} \cdot \mathbf{b} = (2)(1) + (1)(1) + (1)(0) = 3$

Step 2: $|\mathbf{a}| = \sqrt{4 + 1 + 1} = \sqrt{6}$, $|\mathbf{b}| = \sqrt{1 + 1} = \sqrt{2}$

Step 3: $\cos \theta = (\mathbf{a} \cdot \mathbf{b}) / (|\mathbf{a}||\mathbf{b}|) = 3 / (\sqrt{6} \times \sqrt{2}) = 3 / \sqrt{12} = 3 / (2\sqrt{3}) = \sqrt{3} / 2$

Answer: $\theta = 30^\circ$

B2. Show that the vectors $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} - 3\mathbf{j} - 5\mathbf{k}$ and $\mathbf{c} = 3\mathbf{i} - 4\mathbf{j} - 4\mathbf{k}$ form a right-angled triangle.

Step 1: $\mathbf{a} + \mathbf{b} = (2 + 1)\mathbf{i} + (-1 - 3)\mathbf{j} + (1 - 5)\mathbf{k} = 3\mathbf{i} - 4\mathbf{j} - 4\mathbf{k} = \mathbf{c}$. Since $\mathbf{a} + \mathbf{b} = \mathbf{c}$, the three vectors form the sides of a triangle.

Step 2: $\mathbf{a} \cdot \mathbf{b} = (2)(1) + (-1)(-3) + (1)(-5) = 2 + 3 - 5 = 0$, so $\mathbf{a}$ is perpendicular to $\mathbf{b}$.

Step 3 (check): $|\mathbf{a}|^2 = 4 + 1 + 1 = 6$, $|\mathbf{b}|^2 = 1 + 9 + 25 = 35$, $|\mathbf{c}|^2 = 9 + 16 + 16 = 41$. Since $6 + 35 = 41$, $|\mathbf{a}|^2 + |\mathbf{b}|^2 = |\mathbf{c}|^2$.

Answer: The triangle is right-angled; the right angle is between the sides $\mathbf{a}$ and $\mathbf{b}$, and $\mathbf{c}$ is the hypotenuse.

B3. Find the value of λ for which the vector $\mathbf{a} + \lambda\mathbf{b}$ is perpendicular to $\mathbf{c}$, where $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\mathbf{c} = \mathbf{i} + \mathbf{j} + \mathbf{k}$.

Step 1: $\mathbf{a} + \lambda\mathbf{b} = (1 + 2\lambda)\mathbf{i} + (2 - \lambda)\mathbf{j} + (3 + \lambda)\mathbf{k}$

Step 2: For perpendicular vectors, $(\mathbf{a} + \lambda\mathbf{b}) \cdot \mathbf{c} = 0$:

$$(1 + 2\lambda)(1) + (2 - \lambda)(1) + (3 + \lambda)(1) = 0$$

Step 3: $6 + 2\lambda = 0 \Rightarrow \lambda = -3$

Check: For $\lambda = -3$, $\mathbf{a} + \lambda\mathbf{b} = -5\mathbf{i} + 5\mathbf{j} + 0\mathbf{k}$, and its dot product with $\mathbf{c}$ is $-5 + 5 + 0 = 0$.

Answer: $\lambda = -3$

B4. Find the scalar projection and the vector projection of $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$ on $\mathbf{b} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$.

Step 1: $\mathbf{a} \cdot \mathbf{b} = (2)(1) + (-1)(-2) + (1)(2) = 2 + 2 + 2 = 6$

Step 2: $|\mathbf{b}| = \sqrt{1 + 4 + 4} = 3$

Step 3: Scalar projection $= (\mathbf{a} \cdot \mathbf{b}) / |\mathbf{b}| = 6/3 = 2$

Step 4: Vector projection $= [(\mathbf{a} \cdot \mathbf{b}) / |\mathbf{b}|^2] \mathbf{b} = (6/9)(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}) = (2/3)(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$

Answer: Scalar projection = 2; vector projection = $(2/3)\mathbf{i} - (4/3)\mathbf{j} + (4/3)\mathbf{k}$

B5. A force $\mathbf{F} = 3\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$ newtons displaces a particle from the point A(1, 2, 3) to the point B(5, 4, 1). Find the work done by the force.

Step 1: Displacement $\mathbf{d} = \mathbf{AB} = (5 - 1)\mathbf{i} + (4 - 2)\mathbf{j} + (1 - 3)\mathbf{k} = 4\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$

Step 2: Work done $W = \mathbf{F} \cdot \mathbf{d}$

Step 3: $W = (3)(4) + (2)(2) + (-4)(-2) = 12 + 4 + 8 = 24$

Answer: $W = 24$ joules (taking displacement in metres)

B6. Prove that $|\mathbf{a} + \mathbf{b}|^2 + |\mathbf{a} - \mathbf{b}|^2 = 2(|\mathbf{a}|^2 + |\mathbf{b}|^2)$. (Parallelogram law)

Step 1: $|\mathbf{a} + \mathbf{b}|^2 = (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) = \mathbf{a} \cdot \mathbf{a} + 2\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{b} = |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2\mathbf{a} \cdot \mathbf{b}$

Step 2: $|\mathbf{a} - \mathbf{b}|^2 = (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\mathbf{a} \cdot \mathbf{b}$

Step 3: Adding the two results, the terms $\pm 2\mathbf{a} \cdot \mathbf{b}$ cancel:

$$|\mathbf{a} + \mathbf{b}|^2 + |\mathbf{a} - \mathbf{b}|^2 = 2|\mathbf{a}|^2 + 2|\mathbf{b}|^2 = 2(|\mathbf{a}|^2 + |\mathbf{b}|^2)$$

Answer: Hence proved. In a parallelogram, the sum of the squares of the diagonals equals the sum of the squares of the four sides.

B7. Using vectors, prove that the diagonals of a rhombus are perpendicular to each other.

Step 1: Let OABC be a rhombus with $\mathbf{OA} = \mathbf{a}$ and $\mathbf{OC} = \mathbf{b}$. All sides are equal, so $|\mathbf{a}| = |\mathbf{b}|$.

Step 2: Diagonal $\mathbf{OB} = \mathbf{OA} + \mathbf{AB} = \mathbf{a} + \mathbf{b}$ (since $\mathbf{AB} = \mathbf{OC} = \mathbf{b}$). Diagonal $\mathbf{AC} = \mathbf{OC} - \mathbf{OA} = \mathbf{b} - \mathbf{a}$.

Step 3: $(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{b} - \mathbf{a}) = \mathbf{a} \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} - \mathbf{b} \cdot \mathbf{a} = |\mathbf{b}|^2 - |\mathbf{a}|^2$

Step 4: Since $|\mathbf{a}| = |\mathbf{b}|$, this equals 0.

Answer: $\mathbf{OB} \cdot \mathbf{AC} = 0$, so the diagonals $\mathbf{OB}$ and $\mathbf{AC}$ are perpendicular. Hence proved.

B8. If $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$, $|\mathbf{a}| = 3$, $|\mathbf{b}| = 5$ and $|\mathbf{c}| = 7$, find the angle between $\mathbf{a}$ and $\mathbf{b}$.

Step 1: $\mathbf{a} + \mathbf{b} = -\mathbf{c}$, so $|\mathbf{a} + \mathbf{b}|^2 = |\mathbf{c}|^2$

Step 2: $|\mathbf{a}|^2 + |\mathbf{b}|^2 + 2\mathbf{a} \cdot \mathbf{b} = |\mathbf{c}|^2 \Rightarrow 9 + 25 + 2\mathbf{a} \cdot \mathbf{b} = 49 \Rightarrow \mathbf{a} \cdot \mathbf{b} = 15/2$

Step 3: $\cos \theta = (\mathbf{a} \cdot \mathbf{b}) / (|\mathbf{a}||\mathbf{b}|) = (15/2) / (3 \times 5) = 1/2$

Answer: $\theta = 60^\circ$

B9. If $|\mathbf{a}| = 2$, $|\mathbf{b}| = 3$ and $\mathbf{a} \cdot \mathbf{b} = 3$, find $|\mathbf{a} + \mathbf{b}|$, $|\mathbf{a} - \mathbf{b}|$ and the angle between $\mathbf{a}$ and $\mathbf{b}$.

Step 1: $|\mathbf{a} + \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2\mathbf{a} \cdot \mathbf{b} = 4 + 9 + 6 = 19$

Step 2: $|\mathbf{a} - \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\mathbf{a} \cdot \mathbf{b} = 4 + 9 - 6 = 7$

Step 3: $\cos \theta = 3 / (2 \times 3) = 1/2$

Answer: $|\mathbf{a} + \mathbf{b}| = \sqrt{19}$, $|\mathbf{a} - \mathbf{b}| = \sqrt{7}$, $\theta = 60^\circ$

B10. Using vectors, prove the cosine rule $c^2 = a^2 + b^2 - 2ab \cos C$ for a triangle ABC.

Step 1: Let $\mathbf{BC} = \mathbf{p}$, $\mathbf{CA} = \mathbf{q}$ and $\mathbf{AB} = \mathbf{r}$, with $|\mathbf{p}| = a$, $|\mathbf{q}| = b$ and $|\mathbf{r}| = c$. Going round the triangle, $\mathbf{p} + \mathbf{q} + \mathbf{r} = \mathbf{0}$, so $\mathbf{r} = -(\mathbf{p} + \mathbf{q})$.

Step 2: $|\mathbf{r}|^2 = |\mathbf{p} + \mathbf{q}|^2 = |\mathbf{p}|^2 + |\mathbf{q}|^2 + 2\mathbf{p} \cdot \mathbf{q}$

Step 3: The vectors $\mathbf{p}$ (B to C) and $\mathbf{q}$ (C to A) follow each other, so the angle between them is $180^\circ - C$. Therefore $\mathbf{p} \cdot \mathbf{q} = ab \cos(180^\circ - C) = -ab \cos C$.

Step 4: Substituting: $c^2 = a^2 + b^2 - 2ab \cos C$.

Answer: Hence proved.

B11. Find the angle between a diagonal of a cube and one of its edges.

Step 1: Take a cube of side 1 with one vertex at the origin and three edges along the axes. The diagonal from the origin is $\mathbf{d} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and one edge is $\mathbf{e} = \mathbf{i}$.

Step 2: $\mathbf{d} \cdot \mathbf{e} = 1$, $|\mathbf{d}| = \sqrt{3}$, $|\mathbf{e}| = 1$

Step 3: $\cos \theta = 1 / \sqrt{3}$

Answer: $\theta = \cos^{-1}(1/\sqrt{3}) \approx 54.74^\circ$

B12. Find the direction cosines and the direction angles of the vector $\mathbf{a} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

Step 1: $|\mathbf{a}| = \sqrt{4 + 4 + 1} = 3$

Step 2: $l = 2/3, m = -2/3, n = 1/3$

Step 3 (check): $l^2 + m^2 + n^2 = 4/9 + 4/9 + 1/9 = 1$

Step 4: $\alpha = \cos^{-1}(2/3) \approx 48.19^\circ, \beta = \cos^{-1}(-2/3) \approx 131.81^\circ, \gamma = \cos^{-1}(1/3) \approx 70.53^\circ$

Answer: Direction cosines: $2/3, -2/3, 1/3$

Section C: Practice Questions (with Answers)

Solve these on your own using the methods shown above, then check your answers.

C1. Find $\mathbf{a} \cdot \mathbf{b}$ if $\mathbf{a} = 4\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = 3\mathbf{i} + 2\mathbf{j} + 5\mathbf{k}$.

Answer: 20

C2. Find the magnitude and the unit vector of $\mathbf{a} = 6\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$.

Answer: $|\mathbf{a}| = 7$; unit vector = $(6\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) / 7$

C3. Find λ if $\lambda\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ is perpendicular to $3\mathbf{i} - \mathbf{j} + 4\mathbf{k}$.

Answer: $\lambda = -2/3$

C4. Find the angle between $\mathbf{a} = \mathbf{i} - \mathbf{j}$ and $\mathbf{b} = \mathbf{j} - \mathbf{k}$.

Answer: 120°

C5. Find the projection of $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ on $\mathbf{b} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$.

Answer: $5/3$

C6. Find the work done by the force $\mathbf{F} = 4\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$ in moving a particle from $A(2, 1, -1)$ to $B(5, 3, 2)$.

Answer: 12 units

C7. If $|\mathbf{a}| = |\mathbf{b}| = |\mathbf{a} + \mathbf{b}| = 1$, find $|\mathbf{a} - \mathbf{b}|$.

Answer: $\sqrt{3}$

C8. If $\mathbf{a} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, show that $\mathbf{a} + \mathbf{b}$ is perpendicular to $\mathbf{a} - \mathbf{b}$.

Answer: $|\mathbf{a}| = |\mathbf{b}| = 3$, so $(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = |\mathbf{a}|^2 - |\mathbf{b}|^2 = 0$

C9. Find the direction cosines of the vector joining $A(1, 2, 3)$ and $B(3, 5, 9)$.

Answer: $2/7, 3/7, 6/7$

C10. If $\mathbf{a} \cdot \mathbf{b} = 0$, $|\mathbf{a}| = 5$ and $|\mathbf{b}| = 12$, find $|\mathbf{a} + \mathbf{b}|$.

Answer: 13

Exam Tips

- Write the formula first, then substitute the values, then simplify.

- For perpendicular vectors use $\mathbf{a} \cdot \mathbf{b} = 0$. For the angle between vectors use $\cos \theta = (\mathbf{a} \cdot \mathbf{b}) / (|\mathbf{a}||\mathbf{b}|)$.
- The value of $\cos \theta$ must lie between -1 and 1 . If it does not, recheck the calculation.
- A positive dot product means an acute angle; a negative dot product means an obtuse angle.
- Give the units for work done (joules) and state the final answer clearly.

Poly Notes Hub