

Applied Physics 1

Chapter: Physical World, Units and Measurements

Dimensional Analysis Set 4: Broad Numerical Problems with Solutions

For Engineering Students (Long Answer, Exam Oriented)

Method used in every derivation: (1) assume the quantity is a product of powers of the variables it depends on, with a dimensionless constant k ; (2) write the dimensions of each variable in terms of M, L, T ; (3) equate the powers of M, L and T on both sides; (4) solve the equations and write the formula. The value of k must be found by experiment.

Basic dimensions: mass $[M]$, length $[L]$, time $[T]$, temperature $[\theta]$, electric current $[A]$, amount of substance $[\text{mol}]$.

A. Dimensional Formulae from Definitions

Q1. Find the dimensional formulae of impulse, power, pressure, torque, angular momentum, surface tension, coefficient of viscosity, moment of inertia and gravitational potential from their definitions.

Quantity	Definition	Working	Result
Impulse	$J = F \times t$	$[M L T^{-2}][T]$	$[M L T^{-1}]$
Power	$P = W / t$	$[M L^2 T^{-2}] / [T]$	$[M L^2 T^{-3}]$
Pressure	$P = F / A$	$[M L T^{-2}] / [L^2]$	$[M L^{-1} T^{-2}]$
Torque	$\tau = r \times F$	$[L][M L T^{-2}]$	$[M L^2 T^{-2}]$
Angular momentum	$L = r \times p$	$[L][M L T^{-1}]$	$[M L^2 T^{-1}]$
Surface tension	$S = F / l$	$[M L T^{-2}] / [L]$	$[M T^{-2}]$
Coefficient of viscosity	$\eta = F / (A dv/dx)$	$[M L T^{-2}] / ([L^2][T^{-1}])$	$[M L^{-1} T^{-1}]$
Moment of inertia	$I = m r^2$	$[M][L^2]$	$[M L^2]$
Gravitational potential	$V = W / m$	$[M L^2 T^{-2}] / [M]$	$[L^2 T^{-2}]$

Note: for viscosity, the velocity gradient $dv/dx = [L T^{-1}] / [L] = [T^{-1}]$.

Q2. Using the heat equations, find the dimensional formulae of specific heat, latent heat, thermal conductivity, coefficient of linear expansion and the universal gas constant. (Take temperature as $[\theta]$.)

Quantity	Equation used	Working	Result
Specific heat s	$Q = m s \Delta\theta$	$[M L^2 T^{-2}] / ([M][\theta])$	$[L^2 T^{-2} \theta^{-1}]$
Latent heat L	$Q = m L$	$[M L^2 T^{-2}] / [M]$	$[L^2 T^{-2}]$
Thermal conductivity K	$Q = K A \Delta\theta t / d$	$[M L^2 T^{-2}][L] / ([L^2][\theta][T])$	$[M L T^{-3} \theta^{-1}]$
Linear expansion α	$\Delta l = \alpha l \Delta\theta$	$[L] / ([L][\theta])$	$[\theta^{-1}]$
Gas constant R	$P V = n R \Delta\theta$	$[M L^{-1} T^{-2}][L^3] / ([\text{mol}][\theta])$	$[M L^2 T^{-2} \text{mol}^{-1} \theta^{-1}]$

B. Systems of Units and Conversion

Q3. A new system of units has length 2 m, mass 4 kg and time 5 s as its units. Find the numerical values of 1 newton, 1 joule, 1 watt and 1 pascal in this system.

Use $n_2 = n_1 [M_1/M_2]^a [L_1/L_2]^b [T_1/T_2]^c$ with $M_1/M_2 = 1/4$, $L_1/L_2 = 1/2$, $T_1/T_2 = 1/5$.

Force $[M L T^{-2}]$: $n_2 = (1/4)^1 (1/2)^1 (1/5)^{-2} = 0.25 \times 0.5 \times 25 = 3.125$.

Energy $[M L^2 T^{-2}]$: $n_2 = (1/4)(1/2)^2 (1/5)^{-2} = 0.25 \times 0.25 \times 25 = 1.5625$.

Power $[M L^2 T^{-3}]$: $n_2 = (1/4)(1/2)^2 (1/5)^{-3} = 0.25 \times 0.25 \times 125 = 7.8125$.

Pressure $[M L^{-1} T^{-2}]$: $n_2 = (1/4)(1/2)^{-1} (1/5)^{-2} = 0.25 \times 2 \times 25 = 12.5$.

Answer: 1 N = 3.125, 1 J = 1.5625, 1 W = 7.8125 and 1 Pa = 12.5 units of the new system.

Q4. The gravitational constant is $G = 6.67 \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2}$. Find its value in a system whose units are 1 km, 1 tonne (1000 kg) and 1 hour.

Step 1: $[G] = [M^{-1} L^3 T^{-2}]$, so $a = -1$, $b = 3$, $c = -2$.

Step 2: $M_1/M_2 = 1 \text{ kg} / 1000 \text{ kg} = 10^{-3}$; $L_1/L_2 = 1 \text{ m} / 1000 \text{ m} = 10^{-3}$; $T_1/T_2 = 1 \text{ s} / 3600 \text{ s}$.

Step 3: $n_2 = 6.67 \times 10^{-11} \times (10^{-3})^{-1} \times (10^{-3})^3 \times (1/3600)^{-2} = 6.67 \times 10^{-11} \times 10^3 \times 10^{-9} \times 1.296 \times 10^7$.

Step 4: $10^3 \times 10^{-9} \times 1.296 \times 10^7 = 12.96$, so $n_2 = 6.67 \times 10^{-11} \times 12.96 = 8.65 \times 10^{-10}$.

Answer: $G = 8.65 \times 10^{-10}$ in the (km, tonne, hour) system.

Q5. In a system, force F, velocity v and time T are taken as fundamental quantities.

Express the dimensions of length, mass, energy, power and density in terms of F, v and T.

Length: $[L] = [v][T] = [v T]$.

Mass: from $F = m a$ with $a = v / T$, $[M] = [F] / [a] = [F][T] / [v] = [F v^{-1} T]$.

Energy: $E = F \times L = [F][v T] = [F v T]$.

Power: $P = E / T = [F v T] / [T] = [F v]$.

Density: $\rho = M / L^3 = [F v^{-1} T] / [v^3 T^3] = [F v^{-4} T^{-2}]$.

Answer: $[L] = [v T]$, $[M] = [F v^{-1} T]$, $[E] = [F v T]$, $[P] = [F v]$, $[\rho] = [F v^{-4} T^{-2}]$.

C. Checking the Correctness of Equations

Q6. Check the dimensional correctness of the following equations and correct the wrong ones.

Equation	LHS	RHS	Result
$F \Delta t = m (v - u)$	$[M L T^{-2}][T] = [M L T^{-1}]$	$[M][L T^{-1}] = [M L T^{-1}]$	Correct
$P = \rho g h$	$[M L^{-1} T^{-2}]$	$[M L^{-3}][L T^{-2}][L] = [M L^{-1} T^{-2}]$	Correct
$\frac{1}{2} m v^2 = m g h$	$[M L^2 T^{-2}]$	$[M][L T^{-2}][L] = [M L^2 T^{-2}]$	Correct
$v = f \lambda$	$[L T^{-1}]$	$[T^{-1}][L] = [L T^{-1}]$	Correct
$E = h v$	$[M L^2 T^{-2}]$	$[M L^2 T^{-1}][T^{-1}] = [M L^2 T^{-2}]$	Correct
$v^2 = u^2 + 2 a t$	$[L^2 T^{-2}]$	$[L^2 T^{-2}]$ and $[L T^{-2}][T] = [L T^{-1}]$	Wrong; correct form is $v^2 = u^2 + 2 a s$
$F = m v / r$	$[M L T^{-2}]$	$[M][L T^{-1}] / [L] = [M T^{-1}]$	Wrong; correct form is $F = m v^2 / r$

Q7. A travelling wave is $y = A \sin(\omega t - k s)$, where y and s are in metre and t in second. Find the dimensions of A, ω , k and of the wave speed ω / k .

The arguments ωt and $k s$ of the sine must be dimensionless.

$[\omega t] = 1$, so $[\omega] = [T^{-1}]$. $[k s] = 1$, so $[k] = [L^{-1}]$. $[A] = [y] = [L]$.

$[\omega / k] = [T^{-1}] / [L^{-1}] = [L T^{-1}]$, the dimension of velocity.

Answer: $[A] = [L]$, $[\omega] = [T^{-1}]$, $[k] = [L^{-1}]$, and ω / k has the dimensions of speed.

Q8. Show that the Reynolds number $Re = \rho v D / \eta$ and the Froude number $Fr = v^2 / (g D)$ are dimensionless. Find Re for water with $\rho = 1000 \text{ kg/m}^3$, $v = 2 \text{ m/s}$, $D = 0.05 \text{ m}$ and $\eta = 1 \times 10^{-3} \text{ Pa s}$.

Reynolds number: $[\rho v D] = [M L^{-3}][L T^{-1}][L] = [M L^{-1} T^{-1}]$ and $[\eta] = [M L^{-1} T^{-1}]$, so Re is dimensionless.

Froude number: $[v^2] = [L^2 T^{-2}]$ and $[g D] = [L T^{-2}][L] = [L^2 T^{-2}]$, so Fr is dimensionless.

Numerical value: $Re = 1000 \times 2 \times 0.05 / (1 \times 10^{-3}) = 100 / 10^{-3} = 1 \times 10^5$.

Answer: Both are dimensionless and $Re = 1 \times 10^5$.

Q9. A quantity is $X = \epsilon_0 L \Delta V / \Delta t$, where ϵ_0 is the permittivity of free space, L a length, ΔV a potential difference and Δt a time interval. Find the quantity that has the same dimensions as X .

Step 1: $[\epsilon_0] = [M^{-1} L^{-3} T^4 A^2]$ (from Coulomb's law).

Step 2: $\Delta V = \text{work} / \text{charge} = [M L^2 T^{-2}] / [A T] = [M L^2 T^{-3} A^{-1}]$.

Step 3: $[X] = [M^{-1} L^{-3} T^4 A^2][L][M L^2 T^{-3} A^{-1}] / [T] = [M^0 L^0 T^1 A^1] / [T] = [A]$.

Answer: X has the dimensions of electric current.

D. Derivation of Formulae Using Dimensions

Q10. The frequency f of oscillation of a liquid drop depends on its density ρ , radius r and surface tension S . Derive the expression for f .

Step 1: Assume $f = k \rho^a r^b S^c$.

Step 2: $[T^{-1}] = [M L^{-3}]^a [L]^b [M T^{-2}]^c = [M^{a+c} L^{-3a+b} T^{-2c}]$.

Step 3: Mass: $a + c = 0$. Length: $-3a + b = 0$. Time: $-2c = -1$, so $c = 1/2$.

Step 4: $a = -1/2$ and $b = 3a = -3/2$.

Answer: $f = k \sqrt{S / (\rho r^3)}$ (Rayleigh's formula).

Q11. The time period T of a satellite depends on the orbital radius r , the mass M of the planet and the gravitational constant G . Derive the expression for T .

Step 1: Assume $T = k r^a M^b G^c$, with $[G] = [M^{-1} L^3 T^{-2}]$.

Step 2: $[T] = [L]^a [M]^b [M^{-1} L^3 T^{-2}]^c = [M^{b-c} L^{a+3c} T^{-2c}]$.

Step 3: Time: $-2c = 1$, so $c = -1/2$. Mass: $b - c = 0$, so $b = -1/2$. Length: $a + 3c = 0$, so $a = 3/2$.

Answer: $T = k \sqrt{r^3 / (G M)}$. Experiment gives $k = 2\pi$, which is Kepler's third law.

Q12. The volume of liquid flowing per second through a narrow tube depends on the pressure gradient P/l , the viscosity η and the radius r . Derive the formula.

Step 1: Assume $V/t = k (P/l)^a \eta^b r^c$. Dimensions: $[V/t] = [L^3 T^{-1}]$, $[P/l] = [M L^{-2} T^{-2}]$, $[\eta] = [M L^{-1} T^{-1}]$.

Step 2: $[L^3 T^{-1}] = [M^{a+b} L^{-2a-b+c} T^{-2a-b}]$.

Step 3: Mass: $a + b = 0$. Time: $-2a - b = -1$. Substituting $b = -a$ gives $-a = -1$, so $a = 1$ and $b = -1$.

Step 4: Length: $-2(1) - (-1) + c = 3$, so $c = 4$.

Answer: $V/t = k P r^4 / (\eta l)$. Experiment gives $k = \pi/8$ (Poiseuille's formula).

Q13. Derive the speed of sound in a gas in terms of the pressure P and density ρ .

Step 1: Assume $v = k P^a \rho^b$. $[L T^{-1}] = [M L^{-1} T^{-2}]^a [M L^{-3}]^b = [M^{a+b} L^{-a-3b} T^{-2a}]$.

Step 2: Time: $-2a = -1$, so $a = 1/2$. Mass: $a + b = 0$, so $b = -1/2$. Length check: $-1/2 + 3/2 = 1$, which matches.

Answer: $v = k\sqrt{(P/\rho)}$ (Newton's formula; Laplace's correction replaces k by $\sqrt{\gamma}$).

Q14. The drag force F on a body moving through air depends on the air density ρ , the cross-sectional area A and the speed v . Derive the expression.

Step 1: Assume $F = k \rho^a A^b v^c$. $[M L T^{-2}] = [M L^{-3}]^a [L^2]^b [L T^{-1}]^c = [M^a L^{-3a+2b+c} T^{-c}]$.

Step 2: Mass: $a = 1$. Time: $-c = -2$, so $c = 2$. Length: $-3 + 2b + 2 = 1$, so $b = 1$.

Answer: $F = k \rho A v^2$. In practice $k = \frac{1}{2} C_d$, where C_d is the drag coefficient.

Q15. The radius R of the shock wave from an explosion depends on the energy E released, the air density ρ and the time t after the explosion. Derive R and estimate it for $E = 4.0 \times 10^{13}$ J, $\rho = 1.2$ kg/m³ and $t = 0.025$ s (take $k = 1$).

Step 1: Assume $R = k E^a \rho^b t^c$. $[L] = [M L^2 T^{-2}]^a [M L^{-3}]^b [T]^c = [M^{a+b} L^{2a-3b} T^{-2a+c}]$.

Step 2: Mass: $b = -a$. Length: $2a + 3a = 1$, so $a = 1/5$ and $b = -1/5$. Time: $c = 2a = 2/5$.

Step 3: $R = k (E t^2 / \rho)^{1/5}$.

Step 4 (numerical): $E t^2 / \rho = 4.0 \times 10^{13} \times (0.025)^2 / 1.2 = 4.0 \times 10^{13} \times 6.25 \times 10^{-4} / 1.2 = 2.08 \times 10^{10}$.

The fifth root: $\log(2.08 \times 10^{10}) = 10.318$, divided by 5 = 2.064, and $10^{2.064} = 116$.

Answer: $R = k (E t^2 / \rho)^{1/5}$; for the given data R is about 1.2×10^2 m (116 m).

E. Natural (Planck) Units from G , h and c

Q16. Using the gravitational constant G , Planck's constant h and the speed of light c , derive expressions for a length, a time and a mass, and find their values. ($G = 6.67 \times 10^{-11}$ SI, $h = 6.63 \times 10^{-34}$ J s, $c = 3 \times 10^8$ m/s; take $k = 1$.)

Dimensions: $[G] = [M^{-1} L^3 T^{-2}]$, $[h] = [M L^2 T^{-1}]$, $[c] = [L T^{-1}]$. Assume $Q = G^a h^b c^d$.

The product has dimensions $[M^{-a+b} L^{3a+2b+d} T^{-2a-b-d}]$.

Length: need M : $-a + b = 0$, L : $3a + 2b + d = 1$, T : $-2a - b - d = 0$. So $b = a$, $d = -3a$, and $3a + 2a - 3a = 2a = 1$.

$a = 1/2$, $b = 1/2$, $d = -3/2$, so $l_p = \sqrt{(G h / c^3)}$.

Time: need M : $-a + b = 0$, L : $3a + 2b + d = 0$, T : $-2a - b - d = 1$. So $b = a$, $d = -5a$, and $-3a + 5a = 2a = 1$.

$a = 1/2$, $b = 1/2$, $d = -5/2$, so $t_p = \sqrt{(G h / c^5)}$.

Mass: need M : $-a + b = 1$, L : $3a + 2b + d = 0$, T : $-2a - b - d = 0$. Adding the L and T equations gives $a + b = 0$, so with $b - a = 1$ we get $a = -1/2$ and $b = 1/2$, $d = 1/2$.

So $m_p = \sqrt{(h c / G)}$.

Numerical values: $G h = 6.67 \times 10^{-11} \times 6.63 \times 10^{-34} = 4.42 \times 10^{-44}$. $c^3 = 2.7 \times 10^{25}$ and $c^5 = 2.43 \times 10^{42}$. $h c = 1.99 \times 10^{-25}$.

$l_p = \sqrt{(4.42 \times 10^{-44} / 2.7 \times 10^{25})} = \sqrt{(1.64 \times 10^{-69})} = 4.05 \times 10^{-35}$ m.

$t_p = \sqrt{(4.42 \times 10^{-44} / 2.43 \times 10^{42})} = \sqrt{(1.82 \times 10^{-86})} = 1.35 \times 10^{-43}$ s.

$m_p = \sqrt{(1.99 \times 10^{-25} / 6.67 \times 10^{-11})} = \sqrt{(2.98 \times 10^{-15})} = 5.46 \times 10^{-8}$ kg.

Answer: $l_p \approx 4 \times 10^{-35}$ m, $t_p \approx 1.4 \times 10^{-43}$ s, $m_p \approx 5.5 \times 10^{-8}$ kg (with h ; using $h/2\pi$ changes them by a factor of about $\sqrt{(2\pi)}$).

F. Limitations of Dimensional Analysis (Numerical Illustrations)

Q17. Show that $s = u t + a t^2$ is dimensionally correct but physically wrong, and that dimensional analysis cannot derive $s = u t + \frac{1}{2} a t^2$.

Part 1: $[s] = [L]$; $[u t] = [L T^{-1}][T] = [L]$; $[a t^2] = [L T^{-2}][T^2] = [L]$. All terms match, so $s = u t + a t^2$ passes the dimensional test.

But the correct relation is $s = u t + \frac{1}{2} a t^2$. Dimensional analysis cannot see the factor $\frac{1}{2}$, so it cannot reject the wrong equation.

Part 2: Try $s = k u^a t^b a_c^d$, where a_c is the acceleration. $[L] = [L T^{-1}]^a [T]^b [L T^{-2}]^d = [L^{a+d} T^{-a+b-2d}]$.

This gives only two equations for three unknowns: $a + d = 1$ and $-a + b - 2d = 0$.

Choosing $d = 0$ gives $a = 1$, $b = 1$ ($s = k u t$). Choosing $a = 0$ gives $d = 1$, $b = 2$ ($s = k a_c t^2$). Both are valid, so the powers are not unique.

Answer: A sum of terms cannot be derived, the constants (1 and $\frac{1}{2}$) are not given, and a correct-looking equation may still be wrong.

Q18. The period of a pendulum may depend on m , l , g and the angular amplitude θ_0 . Show what dimensional analysis can and cannot tell us.

Step 1: Assume $T = k m^a l^b g^c f(\theta_0)$, where θ_0 is an angle (dimensionless) and f is an unknown function.

Step 2: The dimensions of m , l and g give $a = 0$, $b = 1/2$, $c = -1/2$, so $T = \sqrt{l/g} f(\theta_0)$ times a constant.

Step 3: Since θ_0 is dimensionless, any function $f(\theta_0)$ is allowed, and dimensions cannot determine it.

Answer: Dimensional analysis gives the dependence on l and g and shows that T is independent of m , but it cannot give $f(\theta_0)$ or the constant 2π (f is nearly 1 for small amplitudes).